Spring Semester 2013–2014

MATH31072/MATH41072/MATH61072 *Algebraic Topology*

May/June Examination Solutions and Feedback

General comment. *On the whole people lost marks on this examination because of the parts of questions they couldn't do or did inadequately. There wasn't a lot written which was completely wrong.*

A1. (a) A *geometric simplicial surface* is a finite set K of triangles in some $\mathbb{R}^n$ satisfying the following properties.

(i) The intersection condition: Two triangles in K are either (i) disjoint, (ii) intersect in a common vertex, or (iii) intersect in a common edge.

(ii) The connectivity condition: For each pair of vertices there is a path along edges from one to the other.

(iii) The link condition: For each vertex v, the link of the vertex, i.e. the set of edges opposite v in the triangles containing v, form a simple closed polygon.

[5 marks, bookwork]

(b) An *orientation* of a triangle is a cyclic ordering of the vertices. Two triangles with a common edge are *coherently oriented* if the orientations induced on the common edge are opposite. A simplicial surface is *orientable* if all of the triangles can be oriented so that each pair of triangles with a common edge are coherently oriented.

[3 marks, bookwork]

(c) The statement that this is a topological property means that, given two simplicial spaces K_1 and K_2, if the underlying spaces $|K_1|$ and $|K_2|$ are homeomorphic, then K_1 is orientable if and only if K_2 is orientable.

[2 marks, bookwork]

[**Total: 10 marks**]

[*This was pretty well done apart from some rather confused statements for the last part.*]

A2. A *geometric simplicial complex* is a non-empty finite set K of simplices in some Euclidean space $\mathbb{R}^n$ such that

(i) **the face condition:** if $\sigma \in K$ and $\tau \prec \sigma$ then $\tau \in K$;

(ii) **the intersection condition:** if σ_1 and $\sigma_2 \in K$ then $\sigma_1 \cap \sigma_2 \in K$ and $\sigma_1 \cap \sigma_2 \prec \sigma_1$, $\sigma_1 \cap \sigma_2 \prec \sigma_2$.

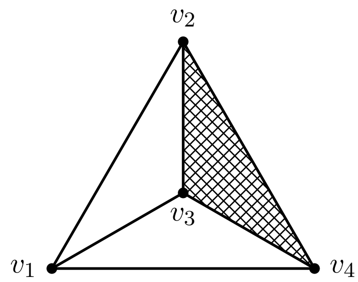
[2 marks, bookwork]

The *underlying space* $|K|$ of a simplicial complex K is given by

$$|K| = \bigcup_{\sigma \in K} \sigma \subset \mathbb{R}^n$$

with the subspace topology. [1 mark, bookwork]

A realization of the given abstract complex as a geometric complex is as follows.



[2 marks, similar to question set]

The *Euler characteristic* of a simplicial complex K is given by the alternating sum

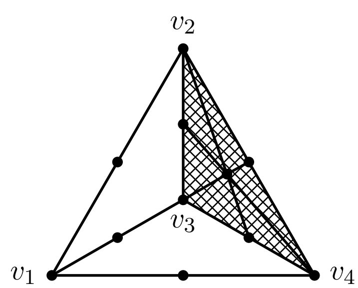
$$\chi(K) = \sum_{r=0}^{\infty} (-1)^r n_r$$

where n_r is the number of simplices of dimension r. [1 mark, bookwork]

In this case, $n_0 = 4$, $n_1 = 6$ and $n_2 = 1$ and so $\chi(K) = 4 - 6 + 1 = -1$.

[1 mark, similar to question set]

The first barycentric subdivision is as follows.



[2 marks, similar to question set]

This also has Euler characteristic -1 since the Euler characteristic is unchanged by barycentric subdivision (or because it is a topological invariant and the underlying space is unchanged) [It can also be found by counting simplices.]. [1 mark, simple application]

[**Total: 10 marks**]

[*Many people got the first barycentric subdivision wrong usually with not enough simplices.*]

A3. For $r \in \mathbb{Z}$. the *r-chain group of K*, denoted $C_r(K)$, is the free abelian group generated by K_r, the set of (non-empty) oriented r-simplices of K subject to the relation $\sigma + \tau = 0$ whenever σ and τ are the same simplex with the opposite orientations. [2 marks, bookwork]

For each $r \in \mathbb{Z}$ we define the *boundary homomorphism* $d_r \colon C_r(K) \to C_{r-1}(K)$ on the generators of $C_r(K)$ by

$$d_r(\langle v_0, v_1, \ldots, v_r \rangle) = \sum_{i=0}^{r} (-1)^i \langle v_0, v_1, \ldots, \hat{v}_i, \ldots, v_r \rangle$$

and then extend linearly. Here $\hat{v}_i$ indicates that this vertex should be omitted.

[2 marks, bookwork]

The kernel of the boundary homomorphism $d_r \colon C_r(K) \to C_{r-1}(K)$ is called the *r-cycle group* of K and is denoted $Z_r(K)$. Thus

$$Z_r(K) = \{\, x \in C_r(K) \mid d_r(x) = 0 \,\}.$$

[1 mark, bookwork]

The image of the boundary homomorphism $d_{r+1} \colon C_{r+1}(K) \to C_r(K)$ is called the *r-boundary group* of K and is denoted $B_r(K)$. Thus

$$B_r(K) = \{\, x \in C_r(K) \mid x = d_{r+1}(y) \text{ for some } y \in C_{r+1}(K) \,\}.$$

[1 mark, bookwork]

In the case of K in Question A.3 we can see that

- $Z_1(K)$ is generated by $x_1 = \langle v_1, v_2\rangle - \langle v_1, v_3\rangle + \langle v_2, v_3\rangle$, $x_2 = \langle v_1, v_2\rangle - \langle v_1, v_3\rangle + \langle v_2, v_3\rangle$ and $x_3 = \langle v_2, v_3\rangle - \langle v_2, v_4\rangle + \langle v_3, v_4\rangle$.
- $B_1(K)$ is generated by x_3.

[2 marks, similar to questions set]

The kernel of the homomorphism $Z_1(K) \to \mathbb{Z}^2$ defined by $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 \mapsto (\lambda_1, \lambda_2)$ has kernel generated by x_3 and so is $B_1(K)$. Hence by the First Isomorphism Theorem this induces an isomorphism $H_1(K) = Z_1(K)/B_1(K) \cong \mathbb{Z}^2$. [2 marks, similar to questions set]

[Total: 10 marks]

[*Those who had covered this material got most of the marks here.*]

A4. (a) The underlying space of $K = \bar{\Delta}^7$ is the 7-simplex Δ^7 which is a convex subset of $\mathbb{R}^8$ and so is contractible. Hence it has the same homology groups as a point:

$$H_i(K) = \begin{cases} \mathbb{Z} & \text{for } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

[3 marks, standard example]

(b) For subcomplex L of K, $n_0 = 8$, $n_1 = \binom{8}{2} = 28$, $n_2 = \binom{8}{3} = 56$ and $n_3 = \binom{8}{4} = 70$ and so the Euler characteristic $\chi(L) = 8 - 28 + 56 - 70 = -34$.

[2 marks, similar to example set]

Now L is 3-dimensional and so and so has trivial homology groups in dimensions above 3. In dimensions $0 \leqslant i \leqslant 3$, $C_i(L) = C_i(K)$ with the same boundary homomorphims between these groups. Hence in dimensions $0 \leqslant i \leqslant 2$, $H_i(L) = H_i(K)$. However, in dimension 3, $B_3(L) = 0$ since $C_4(L) = 0$ and so $H_3(L) = Z_3(L)$ a free group of rank β_3, the third Betti number of L. Now using the formula $\chi(L) = \sum_{i=0}^{3} (-1)^i \beta_i(L)$ we see that $-34 = 1 - \beta_3$ (since $\beta_1 = \beta_2 = 0$) and so $\beta_3 = 35$. Hence

$$H_i(L) = \begin{cases} \mathbb{Z} & \text{for } i = 0, \\ \mathbb{Z}^{35} & \text{for } i = 3, \\ 0 & \text{otherwise.} \end{cases}$$

[5 marks, similar to example set]

[Total: 10 marks]

[*Quite a few people got the sums wrong at the beginning of part (b) forgetting that a 7-simplex has eight vertices. There was some confusion about the Betti numbers and their relationship to the homology groups so this led to confused answers to the last part.*]

B5. (a) To write down a symbol respresenting a topological polygon with edges identified in pairs a letter is assigned to each edge of the polygon, assigning the same letter to two edges if and only if they are to identified. starting at any vertex, write down the letters in sequence going around the boundary, assigning the exponent -1 at the second appearance if the order to the vertices is reversed. [4 marks, bookwork]

(b) The classification theorem states that every closed surface is representatable by one and only one of the following symbols:

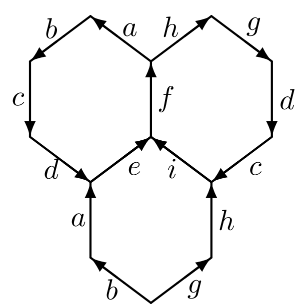
(i) xx^{-1},

(ii) $x_1y_1x_1^{-1}y_1^{-1}x_2y_2x_2^{-1}y_2^{-1}\dots x_gy_gx_g^{-1}y_g^{-1}$ for some integer $g \geqslant 1$,

(iii) $x_1x_1x_2x_2\dots x_gx_g$ for some integer $g \geqslant 1$.

[2 marks, bookwork]

(c) Gluing the discs together along edges e, f and i gives a polygon as follows.



This may be represented by the symbol $abcda^{-1}b^{-1}ghc^{-1}d^{-1}g^{-1}h^{-1}$.

[3 marks, similar to question set]

We may now reduce the symbol corresponding to this identification space to standard form using the standard algorithm as follows.

$$\begin{aligned}
&abcd(a^{-1})(b^{-1}ghc^{-1})d^{-1}g^{-1}h^{-1}\\
&\quad\sim\ a(bcd)(b^{-1}ghc^{-1})a^{-1}d^{-1}g^{-1}h^{-1} \quad (\text{using } \dots xUVx^{-1}\dots \sim \dots xVUx^{-1}\dots)\\
&\quad\sim\ ab^{-1}ghc^{-1}bc(da^{-1}d^{-1})g^{-1}h^{-1} \quad (\text{using } \dots xUVx^{-1}\dots \sim \dots xVUx^{-1}\dots)\\
&\quad\sim\ (ada^{-1}d^{-1})b^{-1}gh(c^{-1}bc)bcg^{-1}h^{-1} \quad (\text{using } xUx^{-1} \text{ commutes with everything})\\
&\quad\sim\ (ada^{-1}d^{-1})(b^{-1}c^{-1}bc)(ghg^{-1}h^{-1}) \quad (\text{using } xUx^{-1} \text{ commutes with everything})\\
&\quad\sim\ x_1y_1x_1^{-1}y_1^{-1}x_2y_2x_2^{-1}y_2^{-1}x_3y_3x_3^{-1}y_3^{-1} \quad (\text{relabelling}).
\end{aligned}$$

[5 marks, similar to question set, standard algorithm]

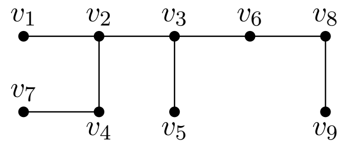
Hence the surface is orientable of genus 3. [1 mark, similar to question set]

[Total: 15 marks]

[*This was reasonably well done on the whole. Some of the answers to (a) were rather confused or not very clear. Some people didn't understand the rules for reducing a symbol to standard form and others who could do the reduction were not very good at explaining what they were doing.*]

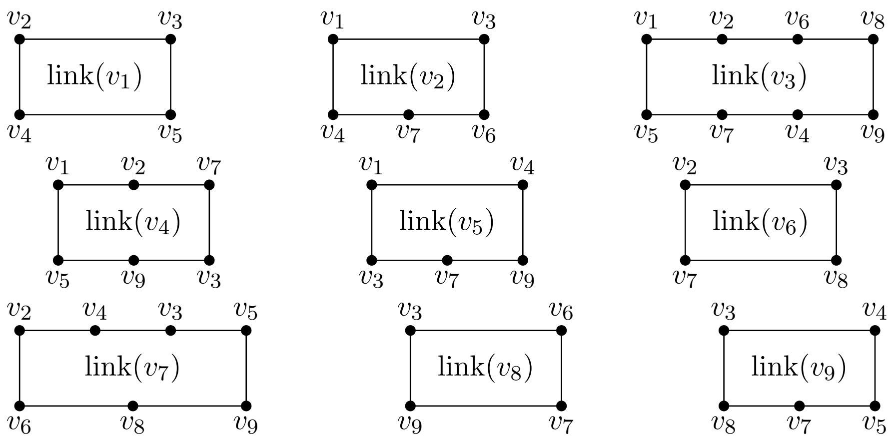
B6. The intersection condition is automatically satisfied since the vertices are linearly independent. [2 marks, standard result from course]

The connectivity condition is satisfied because (for example) of the following edges linking all the vertices.



[1 mark, similar to questions asked]

Checking the link condition we obtain the following.



These are all simple closed polygons.

Hence K is a simplicial surface. [6 marks, similar to questions asked]

$v(K) = 9$, $f(K) = 16$ and $e(K) = 16 \times 3/2 = 24$ (since each edge appears twice).
Hence $\chi(K) = 9 - 24 + 16 = 1$. [2 marks, similar to questions asked]

Since the Euler characteristic of K is odd it is necessarily non-orientable since the Euler characteristic of the oriented surfaces is always even. [2 marks, similar to questions asked]

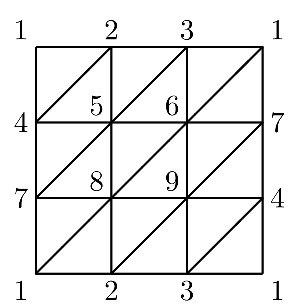
Since $\chi(P_g) = 2 - g$ this means that $|K|$ the underlying space of K, is homeomorphic to P_1, the non-orientable surface of genus 1, i.e. the projective plane.

[2 mark, similar to questions asked]

[Total: 15 marks]

[*You do need to say something about the location of the vertices in order to explain why the intersection condition holds. Checking all nine links is a bit tedious but it is necessary to do them all. It is not necessary to check the definition of orientability since the Euler characteristic is odd.*]

B7. Write v_i for the ith standard basis vector in $\mathbb{R}^9$, $1 \leqslant i \leqslant 9$. Let K be the set of 2-simplices $\langle v_i, v_j, v_k \rangle$ where (i, j, k) are the vertices of a triangle in the triangulation of the unit square I^2 shown below together with their faces. Then K is a simplicial complex with underlying space $|K|$ homeomorphic to the Klein bottle.



The intersection condition is automatic since the vertices are linearly independent vectors and the face condition is automatic by definition.

Now we can define a continuous function $f\colon I^2 \to |K|$ by mapping the point i in the unit square (in the above picture) by $i \mapsto v_i$ and extending linearly over each triangle. This is continuous by the Gluing Lemma (since the triangles are all closed subsets of I^2) and induces a continuous bijection $F\colon I^2/\sim\ \to |K|$ which is therefore a homeomorphism where $\sim$ is the equivalence relation given by $(s,0) \sim (s,1)$ and $(0,t) \sim (1,1-t)$ which is known to give the Klein bottle.

[6 marks, similar to bookwork]

Since K is clearly connected $H_0(K) \cong \mathbb{Z}$ and since K is 2-dimensional $H_i(K) = 0$ for $i > 2$ and $i < 0$.

To find $Z_1(K)$ notice that if $x \in Z_1(K)$ then $x \sim x'$ where x' only involves edges corresponding to the edges of the template together with three 'internal' edges, say $\langle v_4, v_5\rangle$, $\langle v_6, v_7\rangle$ and $\langle v_3, v_9\rangle$. Since other edges can be eliminated. For example $\langle v_2, v_4\rangle \sim \langle v_1, v_4\rangle - \langle v_1, v_2\rangle$ since $d_2\langle v_1, v_2, v_4\rangle = \langle v_2, v_4\rangle - \langle v_1, v_4\rangle + \langle v_1, v_2\rangle \sim 0$. However, since $x \in Z_1(K)$, $x' \in Z_1(K)$ and so x' cannot involve these internal edges since they have vertices which would cancel out on taking the boundary.

Considering the edges corresponding the boundary of the template we see that the cycles containing these edge are generated by $x_1 = \langle v_1, v_2\rangle + \langle v_2, v_3\rangle - \langle v_1, v_3\rangle$ and $x_2 = \langle v_1, v_4\rangle + \langle v_1, v_7\rangle - \langle v_1, \rangle v_7$. Let V be the subgroup of $C_1(K)$ generated by x_1 and x_2. Then $Z_1(K) = V + B_1(K)$.

To find $V \cap B_1(K)$, if $y \in C_2(K)$ such that $d_2(y) \in V$ then the edges corresponding to the internal edges of the template must cancel out and so y must be a multiple of the 2-chain $z = \langle v-1, , v_2, v_4\rangle + \langle v_2, v_5, v_4\rangle + \cdots$ the sum of all of the 2-simplices oriented clockwise in the template.

But $d_2(z) = x_1 - x_2 - x_1 - x_2 = 2x_2$. Hence $V \cap B_1(K) \cong \mathbb{Z}$ generated by $2x_2$.

Thus

$$H_1(K) = Z_1(K)/B_1(K) = \big(V + B_1(K)\big)/B_1(K) \cong V/\big(V \cap B_1(K)\big) \cong \mathbb{Z} \times \mathbb{Z}_2$$

generated by $[x_1]$ and $[x_2]$, since the kernel of the homomorphism $f\colon V \to \mathbb{Z} \times \mathbb{Z}_2$ given by $\lambda_1 x_1 + \lambda_2 x_2 \mapsto (\lambda_1, [\lambda_2]_2)$ is generated by $2x_2$ and so is $V \cap B_1(K)$.

For $H_2(K)$ notice that, by the above argument, if $y \in Z_2(K)$ then, all of the edges corresponding to the internal edges of the template must cancel out and so y is a multiple of z. But $d_2(z) \neq 0$ and so $Z_2(K) = 0$ which means that $H_2(K) = 0$.

Conclusion; $H_i(K) = \begin{cases} \mathbb{Z} & \text{for } i = 0, \\ \mathbb{Z} \times \mathbb{Z}_2 & \text{for } i = 1, \\ 0 & \text{otherwise.} \end{cases}$ [9 marks, standard example]

[**Total: 15 marks**]

[*On the whole the first part of this question was badly done with most people not constructing a homeomorphism. The second part is a standard type of calculation from the course and those who who attempted this question seem to have the basic ideas.*]

B8. (b) A map of simplicial complexes $s : K \to L$ is induced by a map of the vertex sets $s_0\colon V(K) \to V(L)$ so that if $\{v_0, v_1, \ldots, v_r\}$ is an r-simplex of K then $\{f_0(v_0), f_0(v_1), \ldots, f_0(v_r)\}$ is a simplex in L (possibly of lower dimension since f_0 need not be an injection on the vertices of the simplex. Such a map of the vertices may be extended by linearity over the simplices and gives a continuous function $|s| : |K| \to |L|$ by the Gluing Lemma. A function between the underlying spaces which arises in this way is called a simplicial map. [3 marks, bookwork]

We say that a simplicial map $|s|\colon |K| \to |L|$ is a simplicial approximation to a continuous map $f\colon |K| \to |L|$ if, for each point $x \in |K|$, the point $|s|(x)$ belongs to the carrier of $f(x)$ i.e. simplex of L whose interior contains $f(x)$. [2 marks, bookwork]

To see that $|s| \simeq f$, we may define a homotopy $H\colon |K|\times I \to |L|$ by $H(x,t) = (1-t)|s|(x)+tf(x)$. This formula makes sense since both $|s|(x)$ and $f(x)$ lie in the simplex of L whose interior contains $f(x)$ and so, since simplices are convex, the line interval between them does. It is clearly a continuous function. [2 marks, bookwork]

(b) For the particular example, any simplicial approximation to f must map $0 \to 0$ and $1 \to 1$ since the carrier of $f(0) = 0$ is the vertex $\langle 0\rangle$ and the carrier of $f(1) = 1$ is the vertex $\langle 1\rangle$. The only simplicial map $s\colon K \to K$ which does this is the identity map since $s_0(0)$ and $s_0(1/2)$ must be the vertices of a simplex in K and $s_0(1/2)$ and $s_0(1)$ must be the vertices of a simplex of K. But this map is not a simplicial approximation to $f(x) = x^2$ since, for $x = 1/\sqrt{2}$, $f(x) = 1/2$ which has carrier the vertex $\langle 1/2\rangle$. But $|s|(1/\sqrt{2}) = 1/\sqrt{2}$ does not lie in this 0-simplex. [Alternatively, for $1/2 < x < 1/\sqrt{2}$ (only one value is needed), $1/4 < f(x) < 1/2$ and so the carrier of $f(x)$ is the edge $\langle 0, 1/2\rangle$. But for these values of x, $|s|(x) = x > 1/2$ and so does not lie in this simplex.] [4 marks, similar to example set]

However, if K' is the first barycentric subdivision of K, with vertices 0, 1/4, 1/2, 3/4, 1, a simplicial approximation is given by the vertex map $s_0(0) = 0$, $s_0(1/4) = 0$ or $1/2$, $s_0(1/2) = 1/2$, $s_0(3/4) = 1/2$, $s_0(1) = 1$ [two possibilities as indicated], for if $0 < x < 1/\sqrt{2}$ then the carrier of $f(x)$ is the edge $\langle 0, 1/2\rangle$ and $|s|(x)$ lies in this edge (since $1/\sqrt{2} < 3/4$), if $x = 1/\sqrt{2}$ the carrier of $f(x)$ is the vertex $\langle 1/2\rangle$ and $|s|(1/\sqrt{2}) = 1/2$, and finally, if $1/\sqrt{2} < x < 1$ then the carrier of $f(x)$ is the edge $\langle 1/2, 1\rangle$ which does contain $|s|(x)$.

[4 marks, similar to example set]

[Total: 15 marks]

[*There were some good answers to this question. If you hadn't got the idea of the Simplicial Approximation Theorem you probably avoided this question.*]

C9. (a) A *p-symmetry* of a topological surface S is a homeomorphism $f\colon S \to S$ such that its p-fold interate $f^p = f \circ \cdots \circ f = I\colon S \to S$, the identity map but $f \neq I$. It is *free* when $f(x) \neq x$ for all $x \in S$.

The Klein Bottle, P_2 may be obtained from the unit square I^2 with the identifications $(x, 0) \sim (1-x, 1)$ and $(0, y) \sim (1, y)$. For $p > 2$, a free p-symmetry on the Klein bottle is induced by

$$f(x,y) = \begin{cases} (x, y+2/p), & 0 \leqslant y \leqslant (p-2)/p \\ (1-x, y-(p-2)/p), & (p-2)/p \leqslant y \leqslant 1. \end{cases}$$

[5 marks, problem set]

(b) Let U be an open set as in the question. Since S is a surface there is a closed set $A_1 \subset U$ such that $A_1 \cong B^2$. Choose a closed set $A_2 \subset P^2$ such that $A_2 \cong B^2$. Then we can form $S' = S\#P_p$ as the connected sum of S with p copies of P^2 by removing the interiors of the sets $f^i(A_1)$, $0 \leqslant i \leqslant p-1$, from S, taking p copies of P^2 with the interior of A_2 removed and identifying the boundary circles. Then the p-symmetry $f\colon S \to S$ extends to a p-symmetry $f'\colon S' \to S'$ which cyclically permutes the p projective planes. Since f is free so is f'.

[5 marks, problem set, similar to bookwork]

(c) If p divides $g-2$, $g-2 = pr$ and so $g = 2+pr$. Hence, if we apply the construction described in (b) r times to the free symmetry on P_2 described in (a), we obtain a free p-symmetry on P_g. [2 marks, problem set]

(d) If $f\colon S \to S$ is free p-symmetry on a closed surface S then it can be shown that $S_1 = S/x \sim f^i(x)$ is also a closed surface. A triangulation of S_1 induces a triangulation of S with

$v(S) = pv(S_1)$, $e(S) = pe(S_1)$, $f(S) = pf(S_1)$ and so $\chi(S) = p\chi(S_1)$. If $S = P_g$ then $\chi(S) = 2 - g = p\chi(S_1)$ and so $2 - g$ is divisible by p. Thus, if $g - 2$ is not divisible by p there is no free symmetry on P_g. [5 marks, bookwork]

[**Total: 17 marks**]

[*Most people had covered this much of the additional reading. Some people were a bit confused about the construction in (b).*]

C10. (a) A graph is a 1-dimensional simplicial complex. A graph G can be *embedded* in a topological surface S if the realization of G is homeomorphic to a subspace of S. The path-components of the complement of this subspace are called the *regions* of the embedding. If each of the regions is homeomorphic to an open disc then we say that we have a *2-cell embedding.* [3 marks, bookwork]

Given a 2-cell embedding of a connected graph G with v vertices and e edges in a topological surface S we can extend the embedded graph to a triangulation of S as follows. Each region of the embedding together with the adjacent edges and vertices is a topological polygon. We can triangulate this polygon by adding a new vertex in the interior of the region and joining this by edges to the vertices of the polygon. Thus if the polygon has n edges and vertices, we will add one vertex, add n edges and divide the polygon into n triangular regions. So for the new embedded graph G' we obtain by the process $v' - e' + r' = (v+1) - (e+n) + (r+n-1) = v - e + r$. Applying this process to each region of the 2-cell embedding we obtain a triangulation of the surface and so $v - e + r = \chi(S)$. [5 marks, bookwork]

(b) Since each region has at least three edges and each edge is on the boundary of two regions, $2e \geqslant 3r$. Hence $\chi(S) \leqslant v - e + r \leqslant v - e + 2e/3 = v - e/3$. Hence $2e \leqslant 6v - 6\chi(S)$ so that $2e/v \leqslant 6(1 - \chi(S)/v)$. [2 marks, exercise set]

(c) A *colouring* of a graph is the association of function $c\colon V(G) \to A$ to a set A such that, if $\langle v_1, v_2 \rangle$ is an edge of the graph then $c(v_1) \neq c(v_2)$. The *chromatic number* of a graph G is the cardinality of the smallest set A so that it has a colouring $V(G) \to A$. [2 marks, bookwork]

We prove that G must have a vertex of degree $\leqslant N - 1$ so that the result follows by induction on the number of vertices since, if we suppose that graphs with a smaller number of vertices have a chromatic number no greater than N then we can extend a colouring of the graph without this vertex to the whole graph by selecting a colour for this vertex different from the colours of the adjacent vertices. This is certainly possible if $v \leqslant N$ and so we may suppose that $v > N$ which means that $v > x$.

Hence, from the result of (b), the average vertex degree $2e/v \leqslant 6(1 - \chi(S)/v) \leqslant 6(1 - \chi(S)/x)$ (since $\chi(S) \leqslant 0$ and $v > x$ so that $\chi(S)/v \geqslant \chi(S)/x) = x - 1$ (since $x^2 - 7x + 6\chi(S) = 0$ so that $6\big(x - \chi(S)\big) = x^2 - x$) and so G must have a vertex of degree $\leqslant x - 1$ and so of degree $\leqslant N - 1$ as required to carry out the inductive step. [6 marks, bookwork]

[**Total; 18 marks**]

[*Some people didn't seem to have covered this bit of the additional reading. There was a bit of howler in the proof in the notes for the result at the end of (a) which was pretty obvious. However, the howler was faithfully reproduced by quite a few people in spite of not really making sense!*]

C11. (a) A *triangulable pair* of spaces (X, A) is a topological space X with a subspace A such that there is a homeomorphism $h\colon X \to |K|$, the underlying space of a simplicial complex K, with $h(A) = |L|$ the underlying space of a subcomplex L of K. [1 mark, bookwork]

A reduced homology theory assigns to each non-empty triangulable space X a sequence of groups $\tilde{H}_n(X)$ (for $n \in \mathbb{Z}$) and for each continuous map of triangulable spaces $f\colon X \to Y$ a sequence of homomorphisms $f_*\colon \tilde{H}_n(X) \to \tilde{H}_n(Y)$ such that

(i) for continuous functions $f\colon X \to Y$ and $g\colon Y \to Z$, $g_* \circ f_* = (g \circ f)_*\colon \tilde{H}_n(X) \to \tilde{H}_n(Z)$ for all n;

(ii) for the identity map $I\colon X \to X$, $I_* = I\colon \tilde{H}_n(X) \to \tilde{H}_n(X)$ the identity map for all n;

(iii) [homotopy axiom] for homotopic maps $f \simeq g\colon X \to Y$, $f_* = g_*\colon \tilde{H}_n(X) \to \tilde{H}_n(Y)$ for all n;

(iv) [exactness axiom] for any triangulable pair (X, A) there are boundary homomorphisms $\partial\colon \tilde{H}_n(X/A) \to \tilde{H}_{n-1}(A)$ for all n which fit into a long exact sequence

$$\ldots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{q_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \to \ldots$$

and such that for any continuous function of triangulable pairs $f\colon (X, A) \to (Y, B)$ inducing a map of quotient spaces $\bar{f}\colon X/A \to Y/B$ the following diagram commutes for all n;

$$\begin{tikzcd}
\tilde{H}_n(X/A) \arrow[r, "\bar{f}_*"] \arrow[d, "\partial"'] & \tilde{H}_n(Y/B) \arrow[d, "\partial"'] \\
\tilde{H}_{n-1}(A) \arrow[r, "f_*"] & \tilde{H}_{n-1}(B)
\end{tikzcd}$$

(v) [dimension axiom] $\tilde{H}_0(S^0) \cong \mathbb{Z}$ and $\tilde{H}_n(S^0) = 0$ for all $n \neq 0$.

[7 marks, bookwork]

(c) Suppose that $f\colon X \to Y$ is a homotopy equivalence with homotopy inverse $g\colon Y \to X$. Then

$$g_* \circ f_* = (g \circ f)_* \text{ (using (i)) } = I_* \text{ (using (iii)) } = I\colon \tilde{H}_n(X) \to \tilde{H}_n(X) \text{ (by (ii))}$$

and similarly $f_* \circ g_*\colon I\colon \tilde{H}_n(Y) \to \tilde{H}_n(X)$ so that $f_*\colon \tilde{H}_n(X) \to \tilde{H}_n(Y)$ is an isomorphism.

[2 marks, exercise set]

(d) Now consider the pair (D^n, S^{n-1}) for which $D^n/S^{n-1} \cong S^n$. Then the exactness axiom gives the long exact sequence

$$\ldots \to \tilde{H}_i(D^n) \xrightarrow{q_*} \tilde{H}_i(S^n) \xrightarrow{\partial} \tilde{H}_{i-1}(S^{n-1}) \xrightarrow{i_*} \tilde{H}_{i-1}(D^n) \ldots$$

The space D^n is contractible (homotopy equivalent to a point) and so by the above all of its homology groups are trivial. Hence from this exact sequence we see that the boundary homomorphisms

$$\partial\colon \tilde{H}_i(S^n) \to \tilde{H}_{i-1}(S^{n-1})$$

are all isomorphisms. Hence, iterating these maps and using the dimension axiom we see that

$$\tilde{H}_i(S^n) \cong \tilde{H}_{i-n}(S^0) \cong \mathbb{Z} \text{ for } i = n, 0 \text{ for } i \neq n.$$

[5 marks, exercise set]

[Total: 15 marks]

[*Again some people hadn't covered this part of the additional reading.*]

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