Three hours

THE UNIVERSITY OF MANCHESTER

ALGEBRAIC TOPOLOGY

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Answer **ALL** FOUR questions in Section A (40 marks in total). Answer **THREE** of the FOUR questions in Section B (45 marks in total). Answer **ALL** THREE questions in Section C (50 marks in total). If more than THREE questions from Section B are attempted then credit will be given for the best THREE answers.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL** FOUR questions.

A1.

(a) Define what is meant by a *topological manifold*.

(b) State the classification theorem for connected compact topological surfaces.

(c) Give an example of two distinct surfaces from the classification theorem with the same Euler characteristic.

[10 marks]

A2.

(a) Define what is meant by a *geometric simplicial complex* K and its underlying space $|K|$. [The notions of geometric simplex and face of a simplex may be used without definition.]

(b) An abstract simplicial complex has vertices v_1, v_2, v_3, v_4, v_5 and simplices $\{v_1, v_2, v_3\}$, $\{v_2, v_4\}$, $\{v_4, v_5\}$, $\{v_3, v_5\}$, $\{v_2, v_5\}$ and their faces. Draw a realisation K of this simplicial complex as a geometric simplicial complex in $\mathbb{R}^2$.

(c) Define the *Euler characteristic* of a simplicial complex and calculate the Euler characteristic of the simplicial complex in part (b).

(d) Draw the first barycentric subdivision K' of the geometric simplicial complex K in part (b).

(e) Find the Euler characteristic of K'.

[10 marks]

A3.

(a) Define what is meant by the *r-chain group* $C_r(K)$, the *r-cycle group* $Z_r(K)$, and the *r-boundary group* $B_r(K)$ of a simplicial complex K.

(b) Write down, without proof, generators for the groups $Z_1(K)$ and $B_1(K)$ of the simplicial complex K in Question A2(b). Hence, find the first homology group $H_1(K)$.

[10 marks]

A4.

(a) Let $\bar{\Delta}^6$ be be the simplicial complex consisting of all of the faces of the standard 6-simplex in $\mathbb{R}^7$ (including the 6-simplex itself). Consider the simplicial complex K obtained from starring in the barycentre of Δ^6. Write down the simplicial homology groups of K. Justify your answer.

(b) Let L be the 3-skeleton of K. Calculate the Euler characteristic of L and find its simplicial homology groups

[10 marks]

SECTION B

Answer **THREE** of the FOUR questions.

B5. Let e_i be the ith standard basis vector in $\mathbb{R}^8$, $1 \leqslant i \leqslant 8$. Consider the set K of sixteen triangles with vertices e_i , e_j and e_k where ijk runs over the following triples:

$$126, 236, 138, 148, 348, 146, 365, 345, 467, 675, 472, 751, 452, 152, 237, 137.$$

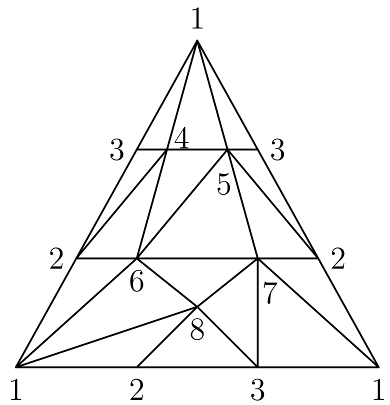
(a) Verify that K is a simplicial surface.
[For the link condition, you need only check the vertices e_1 and e_8 to illustrate the method.]

(b) Represent the underlying space of K as a polygon with edges identified in pairs, and hence represent $|K|$ by a symbol.

(c) Reduce the symbol to canonical form and hence determine the genus and orientability type of the surface $|K|$.

[15 marks]

B6. Consider the triangulation K of the *dunce hat* given by the following picture:



(a) Show that the simplicial homology groups of the dunce hat are given by

$$H_i(K) = \begin{cases} \mathbb{Z} & \text{for } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

You may use the fact, that every 1-cycle is homologous to one involving only edges on the boundary of the template and e.g. the following "internal" edges: $\langle 3, 4\rangle$, $\langle 3, 5\rangle$, $\langle 2, 6\rangle$, $\langle 2, 7\rangle$ and $\langle 1, 8\rangle$.

(b) Use the classification theorem to show that the dunce hat is *not* a closed surface.

[15 marks]

B7.

(a) Define the rth *Betti number* β_r of a simplicial complex K

(b) Define the *Euler characteristic* $\chi(K)$ of a simplicial complex K.

(c) State and prove the relationship between the Betti numbers and the Euler characteristic of K.

(d) Give an application of this relationship.

[15 marks]

B8.

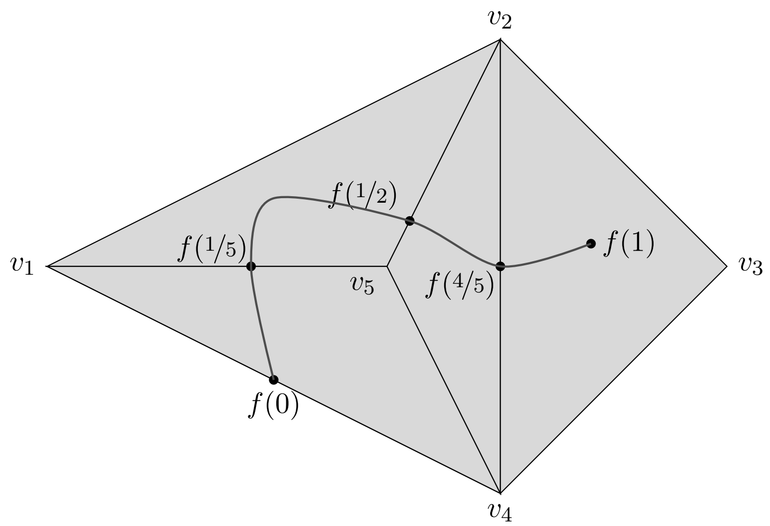
(a) Let K and L be simplicial complexes. Define what is meant by a *simplicial map* $|K| \to |L|$ (with respect to K and L). Define what is meant by a *simplicial approximation* to a continuous map $f\colon |K| \to |L|$ (with respect to K and L).

Prove that a vertex map s with $f(\mathrm{star}(v)) \subset \mathrm{star}(s(v))$ for all vertices v of K induces a simplicial approximation to f.

(b) Consider the simplicial complex L with vertices v_1, v_2, v_3, v_4, v_5, which is drawn below, and an injective continuous map $f\colon [0,1] \to |L|$ with

$$f(0) \in \langle v_1, v_4\rangle,\ f(1/5) \in \langle v_1, v_5\rangle,\ f(1/2) \in \langle v_2, v_5\rangle,\ f(4/5) \in \langle v_2, v_4\rangle,\ f(1) \in \langle v_2, v_3, v_4\rangle$$

and having the image indicated in the picture. Let K be the simplicial complex consisting just of the simplex $\langle 0, 1\rangle$ and its faces. Give a simplicial approximation to f on a sufficiently fine barycentric subdivision $K^{(m)}$ of K.



[15 marks]

SECTION C

Answer **ALL** THREE questions.

C9.

(a) Let p be an odd prime. What is a *p-symmetry* of a topological surface? What is a *fixed point* of such a symmetry?

(b) Given a closed surface X with a p-symmetry f with n fixed points, outline a proof for the identity

$$\chi(S) = p \cdot \chi(S/\sim) - n(p-1).$$

Here, $S/\sim$ is assumed to be a closed surface obtained as a quotient via the equivalence relation $\sim$ defined by $f^i(x) \sim x$ for $i = 1, \ldots, p$.

(c) Use (b) to determine which closed surfaces S admit a p-symmetry with a finite number of fixed points, such that S is homeomorphic to the quotient $S/\sim$.

[17 marks]

C10.

(a) Show that if a graph G with v vertices and e edges embeds in a surface of Euler characteristic χ then $\chi \leqslant v - e/3$.

(b) Show that in the situation above for a complete graph G the inequality $v^2 - 7v + 6\chi \leqslant 0$ holds.

(c) Prove that if a complete graph with v vertices embeds into a surface of Euler characteristic $\chi = \frac{7v-v^2}{6}$ then this embedding is 2-cell and every region is bounded by exactly three edges.

(d) Show that the projective plane has a *unique* minimal triangulation.

[17 marks]

C11.

(a) Define what is meant by saying that (X, A) is a *triangulable pair* of spaces.

(b) State the axioms for the reduced homology groups of triangulable spaces.

(c) Prove that if there is a short exact sequence of abelian groups

$$0 \to H \xrightarrow{i} G \xrightarrow{q} \mathbb{Z} \to 0$$

then $G \cong H \times \mathbb{Z}$.

(d) Determine the reduced homology groups of the disjoint union of a triangulable space X and a single point in terms of the groups $\tilde{H}_i(X)$, $i \geqslant 0$.

[16 marks]

END OF EXAMINATION PAPER