SECTION A

Answer **ALL** FOUR questions.

A1.

(a) Define what is meant by a *topological manifold*.

(b) State the classification theorem for connected compact topological surfaces.

(c) Give an example of two distinct surfaces from the classification theorem with the same Euler characteristic.

[10 marks]

Solution

(a) Let n be a non-negative integer. An *n-dimensional (topological) manifold* is a topological space X which

(i) is Hausdorff,

(ii) is second countable (i.e. has a countable basis), and

(iii) is *locally Euclidean*, i.e. each point $x \in X$ lies in an open subset V in X which is homeomorphic to an open subset $U \subset \mathbb{R}^n$ (with the usual topology).

[5 marks, bookwork]

(b) Every path-connected compact topological surface (or *closed surface*) is homeomorphic to one and only one of:

(i) S^2,

(ii) T_g for some $g \geqslant 1$ (where $T_1 = S^1 \times S^1$ and $T_{g+1} = T_g \# T_1$ for $g \geqslant 1$),

(iii) P_g for some $g \geqslant 1$ (where $P_1 = P^2$ and $P_{g+1} = P_g \# P_1$ for $g \geqslant 1$). [3 marks, bookwork]

(c) $\chi(T_k) = 2 - 2k = \chi(P_{2k}) = 2 - 2k$. Hence, for every choice of $k \geqslant 1$ this gives a suitable example. [2 marks, bookwork]

[Total: 10 marks]

Feedback: The question was meant to be easy. The definition of a manifold was fundamental for the first part of the course (on surfaces). However, quite a few people where not able to reproduce it (correctly). Some people forgot to explain what locally Euclidean means. In part (b) some people mixed up the two classification theorems for surfaces and surface symbols (which of course are very much related to each other). Part (c) was generally done well.

A2.

(a) Define what it meant by a *geometric simplicial complex* K and its underlying space $|K|$.
[The notions of geometric simplex and face of a simplex may be used without definition.]

(b) An abstract simplicial complex has vertices v_1, v_2, v_3, v_4, v_5 and simplices $\{v_1, v_2, v_3\}$, $\{v_2, v_4\}$, $\{v_4, v_5\}$, $\{v_3, v_5\}$, $\{v_2, v_5\}$ and their faces. Draw a realisation K of this simplicial complex as a geometric simplicial complex in $\mathbb{R}^2$.

(c) Define the *Euler characteristic* of a simplicial complex and calculate the Euler characteristic of the simplicial complex in part (b).

(d) Draw the first barycentric subdivision K' of the geometric simplicial complex K in part (b).

(e) Find the Euler characteristic of K'.

[10 marks]

Solution

(a) A *(geometric) simplicial complex* is a non-empty finite set K of simplices in some Euclidean space $\mathbb{R}^n$ such that

(a) **the face condition:** if $\sigma \in K$ and $\tau \prec \sigma$ then $\tau \in K$,

(b) **the intersection condition:** if σ_1 and $\sigma_2 \in K$ then $\sigma_1 \cap \sigma_2 = \emptyset$ or $\sigma_1 \cap \sigma_2 \prec \sigma_1$, $\sigma_1 \cap \sigma_2 \prec \sigma_2$.

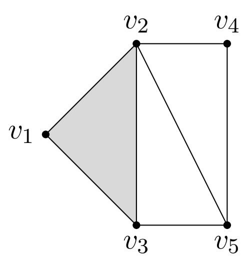
[2 marks, bookwork]

The *underlying space* $|K|$ of a simplicial complex K is given by

$$|K| = \bigcup_{\sigma \in K} \sigma \subset \mathbb{R}^n$$

with the subspace topology. [1 mark, bookwork]

(b) A realisation is given by the following picture



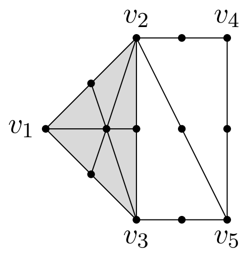
[2 marks, similar to question set]

(c) The *Euler characteristic* of a simplicial complex K is given by the alternating sum

$$\chi(K) = \sum_{r=0}^{\infty} (-1)^r n_r$$

where n_r is the number of simplices of dimension r. In this case $\chi(K) = 5 - 7 + 1 = -1$. [2 marks, bookwork]

(d) The barycentric subdivision is given by the following picture



[2 marks, similar to question set]

(e) The Euler characteristic is again -1, since barycentric subdivisions does not change the Euler characteristic [It can also be found by counting simplices.] [1 mark, simple application]

[Total: 10 marks]

Feedback: The question was generally done well. Some people forgot the definition of the underlying space.

A3.

(a) Define what is meant by the r-*chain group* $C_r(K)$, the r-*cycle group* $Z_r(K)$, and the r-*boundary group* $B_r(K)$ of a simplicial complex K.

(b) Write down, without proof, generators for the groups $Z_1(K)$ and $B_1(K)$ of the simplicial complex K in Question A2(b). Hence, find the first homology group $H_1(K)$.

[10 marks]

Solution

(a) For $r \in \mathbb{Z}$. the r-chain group of K, denoted $C_r(K)$, is the free abelian group generated by K_r, the set of oriented r-simplices of K subject to the relation $\sigma + \tau = 0$ whenever σ and τ are the same simplex with the opposite orientations. [2 marks, bookwork]

For each $r \in \mathbb{Z}$ we define the boundary homomorphism $d_r : C_r(K) \to C_{r1}(K)$ on the generators

$$d_r(\langle v_0, \dots, v_r\rangle) = \sum_{i=0}^{r} (-1)^i \langle v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_r\rangle$$

and then extend linearly. [2 marks, bookwork]

The kernel of the boundary homomorphism d_r is called the r-cycle group and denoted by $Z_r(K)$, i.e. $Z_r(K) = \{c \in C_r(K) \mid d_r(c) = 0\}$. [1 mark, bookwork]

The image of the boundary homomorphism d_{r+1} is called the r-boundary group and is denoted by $B_r(K)$, i.e. $B_r(K) = \{d_{r+1}(c) \mid c \in C_{r+1}(K)\}$. [1 mark, bookwork]

(b) $Z_1(K) \cong \mathbb{Z}^3$ is generated by

$$\begin{aligned} z_1 &= \langle v_1, v_3\rangle + \langle v_3, v_2\rangle + \langle v_2, v_1\rangle \\ z_2 &= \langle v_3, v_5\rangle + \langle v_5, v_2\rangle + \langle v_2, v_3\rangle \\ z_3 &= \langle v_5, v_4\rangle + \langle v_4, v_2\rangle + \langle v_4, v_5\rangle \end{aligned}$$

$B_1(K) \cong \mathbb{Z}$ is generated by z_1. [2 marks, similar to question set]

We obtain

$$H_1(K) = Z_1(K)/B_1(K) = (\mathbb{Z}z_1 \oplus \mathbb{Z}z_2 \oplus \mathbb{Z}z_3)/\mathbb{Z}z_1 \cong \mathbb{Z}^2.$$

[2 marks, similar to question set]

[Total: 10 marks]

Feedback: The question was generally done well. The most common mistake in (a) was a missing reference to the orientation of a simplex. In part (b) I have often seen notation like $\mathbb{Z}^3/\mathbb{Z}$. Note, that this does not make sense, since $\mathbb{Z}$ is not a subgroup of $\mathbb{Z}^3$ (although there are many subgroups of $\mathbb{Z}^3$ being isomorphic to $\mathbb{Z}$ but the quotient will depend on the choice of such a subgroup).

A4.

(a) Let $\bar{\Delta}^6$ be be the simplicial complex consisting of all of the faces of the standard 6-simplex in $\mathbb{R}^7$ (including the 6-simplex itself). Consider the simplicial complex K obtained from starring in the barycentre of Δ^6. Write down the simplicial homology groups of K. Justify your answer.

(b) Let L be the 3-skeleton of K. Calculate the Euler characteristic of L and find its simplicial homology groups

[10 marks]

Solution

(a) The underlying space of K is the 6-simplex Δ^6 which is a convex subset of $\mathbb{R}^7$ and so is contractible. Hence it has the same homology groups as a point: $H_0(K) \cong \mathbb{Z}$ and all other homology groups are trivial.

$$H_i(K) = \begin{cases} \mathbb{Z} & \text{for } i = 0, \\ 0 & \text{else.} \end{cases}$$

[3 marks, standard example]

(b) We have $n_0 = 7 + 1$, $n_1 = \binom{7}{2} + \binom{7}{1}$, $n_2 = \binom{7}{3} + \binom{7}{2}$, $n_3 = \binom{7}{4} + \binom{7}{3}$. By taking the alternating sum we obtain $1 - \binom{7}{4} = -34$. [2 marks, similar to question set]

Now L is 3-dimensional and so and so has trivial homology groups in dimensions above 3. For $0 \leqslant i \leqslant 3$ we have $C_i(K) = C_i(L)$ and the boundary homomorphisms are the same. Hence, $H_i(K) = H_i(L)$ for $0 \leqslant i \leqslant 2$. Since $C_4(L) = 0$ we have $H_3(L) = Z_3(L)$ a free group of rank β_3.

Now, using the identity $-34 = \chi(L) = \sum(-1)^i \beta_i$ we obtain $1 - \beta_3 = -34$(since $\beta_1 = \beta_2 = 0$) and so $\beta_3 = 35$ and $H_3(L) = \mathbb{Z}^{35}$.

$$H_i(L) = \begin{cases} \mathbb{Z} & \text{for } i = 0, \\ \mathbb{Z}^{35} & \text{for } i = 3, \\ 0 & \text{else.} \end{cases}$$

[5 marks, similar to question set]

[Total: 10 marks]

Feedback: Most people used the correct approach to solve the problem. A very common mistake was to assume that the 3-skeleton of the starred simplex is homeomorphic to the 3-skeleton of the simplex, which is wrong (what is true is that a starring of the 3-skeleton of Δ^6 is homeomorphic to the 3-skeleton of Δ^6)

SECTION B

Answer **THREE** of the FOUR questions.

B5. Let e_i be the ith standard basis vector in $\mathbb{R}^8$, $1 \leqslant i \leqslant 8$. Consider the set K of sixteen triangles with vertices e_i , e_j and e_k where ijk runs over the following triples:

$$126, 236, 138, 148, 348, 146, 365, 345, 467, 675, 472, 751, 452, 152, 237, 137.$$

(a) Verify that K is a simplicial surface.
[For the link condition, you need only check the vertices e_1 and e_8 to illustrate the method.]

(b) Represent the underlying space of K as a polygon with edges identified in pairs, and hence represent $|K|$ by a symbol.

(c) Reduce the symbol to canonical form and hence determine the genus and orientability type of the surface $|K|$.

[15 marks]

Solution

(a) The intersection condition is satisfied automatically since the vertices are linearly independent.

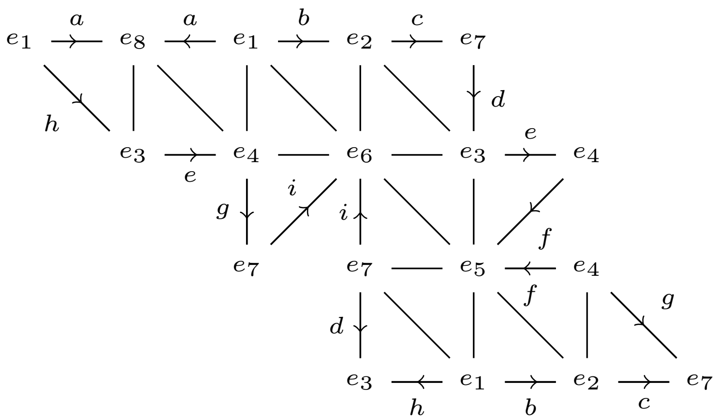
[1 mark]

The connectivity condition is satisfied because (for example) the following edges link all of the vertices: $e_8 - e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7$. [1 mark]

The link of e_1 consists of the edges $e_7 - e_3 - e_8 - e_4 - e_6 - e_2 - e_5 - e_7$, which form a closed simple polygon. The same holds for the link of e_8, which is formed by the edges $e_1 - e_3 - e_4 - e_1$.

[3 marks]

(b) A corresponding polygon with edges identified in pairs might look as follows.



The corresponding symbol is given by

$$aa^{-1}bcdeff^{-1}gc^{-1}b^{-1}hd^{-1}ii^{-1}g^{-1}e^{-1}h^{-1}.$$

[5 marks]

(c) Reducing the symbol to canonical form using the standard alorithm gives

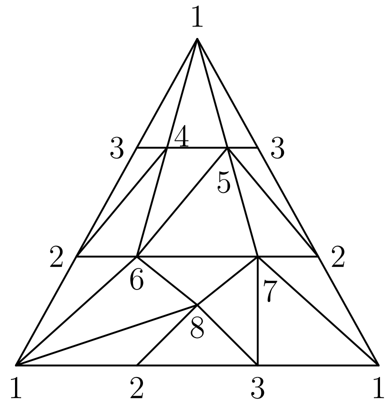
$$
\begin{aligned}
& aa^{-1}bcdeff^{-1}gc^{-1}b^{-1}hd^{-1}ii^{-1}g^{-1}e^{-1}h^{-1} && \\
& \sim b(\dot{c}deg\dot{c}^{-1})b^{-1}hd^{-1}g^{-1}e^{-1}h^{-1} && \text{(cancelling } xx^{-1}\text{)} \\
& \sim bb^{-1}cdegc^{-1}hd^{-1}g^{-1}e^{-1}h^{-1} && \text{(commuting of } cUc^{-1}\text{)} \\
& \sim cde\dot{g}(c^{-1})(hd^{-1})\dot{g}^{-1}e^{-1}h^{-1} && \text{(cancelling } xx^{-1}\text{)} \\
& \sim \dot{c}(deg)(hd^{-1})\dot{c}^{-1}g^{-1}e^{-1}h^{-1} && (gUVg^{-1}... \sim gVUg^{-1}...) \\
& \sim chd^{-1}de(\dot{g}c^{-1}\dot{g}^{-1})e^{-1}h^{-1} && (cUVc^{-1}... \sim cVUc^{-1}...) \\
& \sim cgc^{-1}g^{-1}hd^{-1}dee^{-1}h^{-1} && \text{(commuting of } cUc^{-1}\text{)} \\
& \sim cgc^{-1}g^{-1} && \text{(cancelling } xx^{-1}\text{)}
\end{aligned}
$$

Hence, the surface is orientable of genus 1 (the torus). [5 marks]

[Total: 15 marks, similar to question set]

Feedback: Most people have chosen this question and most did very well.

B6. Consider the triangulation K of the *dunce hat* given by the following picture:



(a) Show that the simplicial homology groups of the dunce hat are given by

$$H_i(K) = \begin{cases} \mathbb{Z} & \text{for } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

You may use the fact, that every 1-cycle is homologous to one involving only edges on the boundary of the template and e.g. the following "internal" edges: $\langle 3,4\rangle$, $\langle 3,5\rangle$, $\langle 2,6\rangle$, $\langle 2,7\rangle$ and $\langle 1,8\rangle$.

(b) Use the classification theorem to show that the dunce hat is *not* a closed surface.

[15 marks]

Solution

(a) To find $Z_1(K)$ first note, that for every 1-cycle x one has $x \sim x'$ for some x' only involving edges corresponding to edges on the boundary of the template and e.g. the following "internal" edges $\langle 3, 4\rangle$, $\langle 3, 5\rangle$, $\langle 2, 6\rangle$, $\langle 2, 7\rangle$ and $\langle 1, 8\rangle$. Since all other edges can be eliminated via boundaries of triangles. However, since $x \in Z_1(K)$ we also have $x' \in Z_1(K)$ and so x' can not involve these internal edges, since their "internal" vertices wouldn't cancel out when taking the boundary.

Consider a cycle

$$x' = \lambda_1\langle 1, 3\rangle + \lambda_2\langle 3, 2\rangle + \lambda_3\langle 2, 1\rangle.$$

Now, $d(x') = 0$ implies $\lambda_1 = \lambda_2 = \lambda_3$. Hence, the subgroup V of cycles involving only edges on the boundary of the template are generated by

$$x = \langle 1, 3\rangle + \langle 3, 2\rangle + \langle 2, 1\rangle.$$

We have $Z_1(K) = V + B_1(K)$. It remains to determine $V \cap B_1(K)$. Consider, some non-trivial cycle $Z_2(K)$. For the inner edges of the template to cancel out when taking the boundary the cycle has to be a multiple of the sum over all triangles (with compatible orientation, e.g. all clockwise orientented), which we denote by y. But then

$$d(\ell y) = \ell\langle 1, 3\rangle + \langle 3, 2\rangle + \langle 2, 1\rangle = \ell x.$$

Hence, $V = Z_1(K) = B_1(K)$ and $H_1(K) = Z_1(K)/B_1(K) = 0$.

For $z \in Z_2(V)$ it must be a multiple of y, but since $d(y) \neq 0$ we have $H_2(K) = Z_2(K) = 0$.

[12 marks, similar to question set]

(b) For the Euler characteristic of the dunce hat we obtain

$$\chi = \beta_0 - \beta_1 + \beta_2 = 1 - 0 + 0 = 1.$$

From the classification theorem we see that the only possible closed surface is the projective plane. On the other hand, we have $H_1(\mathbb{P}^2) = \mathbb{Z}/2\mathbb{Z} \neq 0 = H_1(K)$.

Alternatively, one can argue, that around a point on the identified edges the dunce hat is not locally Euclidean. But, it's harder to actually prove this. [3 marks, new]

[Total: 15 marks]

Feedback: Most people who attempted the question used the correct approach. However, often the given arguments in (a) where not clear (or even wrong), so most people secured partial marks, here. For part (b) one had to be careful with choosing an argument. The Dunce hat is given by identifying edges of a topological polytope, but this time *not* in pairs. However, we didn't learn about a theorem telling us that in this case we don't obtain a surface. Equivalently we didn't learn a theorem telling us that a symbol where the letters do not occur in pairs cannot produce a surface.

B7.

(a) Define the rth *Betti number* β_r of a simplicial complex K

(b) Define the *Euler characteristic* $\chi(K)$ of a simplicial complex K.

(c) State and prove the relationship between the Betti numbers and the Euler characteristic of K.

(d) Give an application of this relationship.

[15 marks]

Solution

(a) The *rth Betti number*. $\beta_r(K)$ of a simplicial complex K is the rank of $H_r(K)$

[2 marks, bookwork]

(b) The *Euler characteristic* of an n-dimensional simplicial complex K is given by $\chi(K) = \sum_{r=0}^{n}(-1)^r n_r$ where n_r is the number of r-simplices.

[2 marks, bookwork]

(c) The relation $H_r(K) = Z_r(K)/B_r(K)$ implies

$$\beta_r(K) = \operatorname{rank}(H_r(K)) = \operatorname{rank}(Z_r(K)) - \operatorname{rank}(B_r(K)).$$

On the other hand one has [2 marks, bookwork]

$$B_{r-1}(K) = \operatorname{Im}(d_r\colon C_r(K) \to C_{r-1}(K))$$

and

$$Z_r(K) = \operatorname{Ker}(d_r\colon C_r(K) \to C_{r-1}(K)).$$

By isomorphism theorem we obtain $C_r(K)/Z_r(K) \cong B_{r-1}(K)$. This implies $\operatorname{rank}(B_{r-1}(K)) = \operatorname{rank}(C_r(K)) - \operatorname{rank}(Z_r(K))$ or equivalently

$$n_r = \operatorname{rank}(C_r(K)) = \operatorname{rank}(B_{r-1}(K)) + \operatorname{rank}(Z_r(K)).$$

[3 marks, bookwork]

Now, we may conclude

$$\begin{aligned}
\chi(K) &= \sum_{r=0}^{n}(-1)^r n_r \quad \text{by definition} \\
&= \sum_{r=0}^{n}(-1)^r \big(\operatorname{rank} Z_r(K) + \operatorname{rank} B_{r-1}(K)\big) \\
&= \sum_{r=0}^{n}(-1)^r \operatorname{rank} Z_r(K) + \sum_{r=0}^{n}(-1)^r \operatorname{rank} B_{r-1}(K) \\
&= \sum_{r=0}^{n}(-1)^r \operatorname{rank} Z_r(K) - \sum_{r=0}^{n}(-1)^r \operatorname{rank} B_r(K) \quad \text{since } B_r(K) = 0 \text{ for } r = -1, n \\
&= \sum_{r=0}^{n}(-1)^r \big(\operatorname{rank} Z_r(K) - \operatorname{rank} B_r(K)\big) \\
&= \sum_{r=0}^{n}(-1)^r \beta_r(K).
\end{aligned}$$

[5 marks, bookwork]

(d) The most important consequence is the homotopy invariance of the Euler characteristic. Another possibility would be the calculation of the homology groups of the n-sphere or even the calculation in A4(b). [1 marks, bookwork]

[Total: 15 marks]

Feedback: Part (a) and (b) didn't cause any trouble. In part (c) one serious mistake was the conclusion that $\sum_r (-1)^r n_r = \sum_r (-1)^r \beta_r$ should imply $n_r = \beta_r$ which is wrong.

B8.

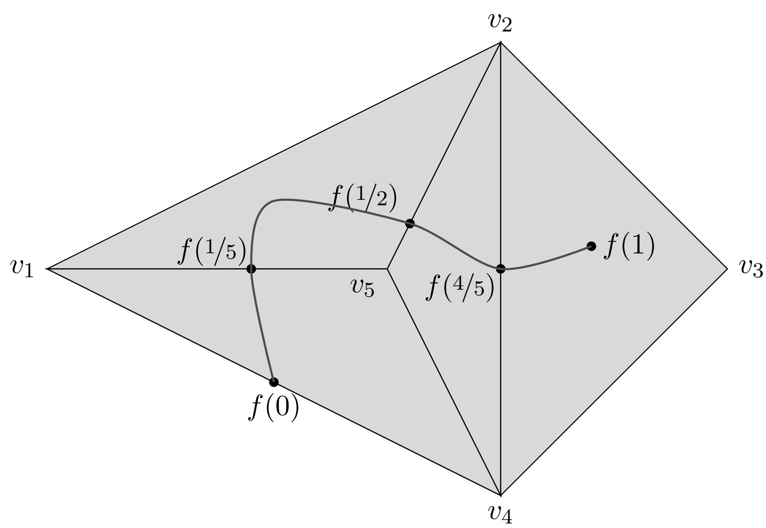
(a) Let K and L be simplicial complexes. Define what is meant by a *simplicial map* $|K| \to |L|$ (with respect to K and L). Define what is meant by a *simplicial approximation* to a continuous map $f\colon |K| \to |L|$ (with respect to K and L).

Prove that a vertex map s with $f(\mathrm{star}(v)) \subset \mathrm{star}(s(v))$ for all vertices v of K induces a simplicial approximation to f.

(b) Consider the simplicial complex L with vertices v_1, v_2, v_3, v_4, v_5, which is drawn below, and an injective continuous map $f\colon [0,1] \to |L|$ with

$$f(0) \in \langle v_1, v_4 \rangle,\ f(1/5) \in \langle v_1, v_5 \rangle,\ f(1/2) \in \langle v_2, v_5 \rangle,\ f(4/5) \in \langle v_2, v_4 \rangle,\ f(1) \in \langle v_2, v_3, v_4 \rangle$$

and having the image indicated in the picture. Let K be the simplicial complex consisting just of the simplex $\langle 0, 1 \rangle$ and its faces. Give a simplicial approximation to f on a sufficiently fine barycentric subdivision $K^{(m)}$ of K.



[15 marks]

Solution

(a) A map of simplicial complexes $s : K \to L$ is induced by a map of the vertex sets $s_0 : V(K) \to V(L)$ so that if $\{v_0, v_1, ..., v_r\}$ is an r-simplex of K then $\{s_0(v_0), s_0(v_1), ..., s_0(v_r)\}$ is a simplex in L (possibly of lower dimension since s_0 need not be an injection on the vertices of the simplex. Such a map of the vertices may be extended by linearity over the simplices and gives a continuous function $|s| : |K| \to |L|$ by the Gluing Lemma. A function between the underlying spaces which arises in this way is called a simplicial map.

[3 marks, bookwork]

We say that a simplicial map $|s| : |K| \to |L|$ is a simplicial approximation to a continuous map $f : |K| \to |L|$ if, for each point $x \in |K|$, the point $|s|(x)$ belongs to the carrier of $f(x)$ i.e. simplex of L whose interior contains $f(x)$.

[2 marks, bookwork]

Given a point x in the interior of $\langle v_0, \ldots, v_r\rangle$ it is contained in $\bigcap_{i=0}^r \mathrm{star}(v_i)$ and, hence,

$$f(x) \in f\left(\bigcap_{i=0}^r \mathrm{star}(v_i)\right) \subset \bigcap_{i=0}^r (\mathrm{star}(s(v_i))$$

In particular, $\bigcap_{i=0}^r(\mathrm{star}(s(v_i))$ is non-empty. If the interior of σ is contained in $\bigcap_{i=0}^r(\mathrm{star}(s(v_i))$ then $s(v_0), \ldots, s(v_r)$ have to be vertices of σ. On the one hand this implies that $\langle s(v_0), \ldots, s(v_r)\rangle$ is a face of σ. In particular, it is a simplex in L. Hence, s is admissible. On the other hand the carrier of every point in $\bigcap_{i=0}^r(\mathrm{star}(s(v_i))$ contains $\langle s(v_0), \ldots, s(v_r)\rangle$ and hence $|s|(x)$.

[3 marks, question set]

(b) We have to take $K^{(2)}$ consisting of the intervals $[0, 1/4]$, $[1/4, 1/2]$, $[1/2, 3/4]$ and $[3/4, 1]$ and their endpoints. Now, one observes that

$$\begin{aligned}
\mathrm{star}(0) &= [0, 1/4) \subset [0, 1/2) = f^{-1}(\mathrm{star}(v_1)),\\
\mathrm{star}(1/4) &= (0, 1/2) \subset (0, 4/5) = f^{-1}(\mathrm{star}(v_5)),\\
\mathrm{star}(1/2) &= (1/4, 3/4) \subset (0, 4/5) = f^{-1}(\mathrm{star}(v_5)),\\
\mathrm{star}(3/4) &= (1/2, 1) \subset [1/5, 1] = f^{-1}(\mathrm{star}(v_2)),\\
\mathrm{star}(1) &= (3/4, 1] \subset [1/5, 1] = f^{-1}(\mathrm{star}(v_2)).
\end{aligned}$$

Hence, by (a) the vertex map s given by $s(0) = v_1$, $s(1/4) = s(1/2) = v_5$ and $s(3/4) = s(1) = v_2$ defines a simplicial approximation to f. [7 marks, similar to question set]

[Total: 15 marks]

Feedback: Only a few people attempted the question.

SECTION C

Answer **ALL** THREE questions.

C9.

(a) Let p be an odd prime. What is a *p-symmetry* of a topological surface? What is a *fixed point* of such a symmetry?

(b) Given a closed surface X with a p-symmetry f with n fixed points, outline a proof for the identity

$$\chi(S) = p \cdot \chi(S/\!\sim) - n(p-1).$$

Here, $S/\!\sim$ is assumed to be a closed surface obtained as a quotient via the equivalence relation $\sim$ defined by $f^i(x) \sim x$ for $i = 1, \dots, p$.

(c) Use (b) to determine which closed surfaces S admit a p-symmetry with a finite number of fixed points, such that S is homeomorphic to the quotient $S/\!\sim$.

[17 marks]

Solution

(a) A p-symmetry of a topological surface S is a homeomorphism $f : S \to S$, such that its p-fold iterate $f^p = f \circ \cdots \circ f = \mathrm{id}_S : S \to S$, the identity map but $f \neq \mathrm{id}_S$. A fixed point of a p-symmetry f is a point $x \in S$, such that $f(x) = x$. [4 marks, bookwork]

(b) By the formulation of the question S' is assumed to be a topological surface (indeed, this is always the case) and, hence, admits a triangulation K'. One can choose the triangulation (e.g. by starring if necessary) such that the images of fixed points become vertices and no two such vertices belong to the same edge. Now, one uses the quotient map q to construct a triangulation K for S, such that the map $|K| \to |K'|$ corresponding to q maps vertices to vertices and triangles to triangles. Then we obtain $e(K') = e(K)$ and $f(K) = f(K')$, but for the numbers of vertices we have $v(K) - n = p \cdot (v(K') - n)$, since the fixed points have only a single preimage under the map q, but all other points have exactly p points in the preimage. Alltogether one obtains

$$\chi(S) = v(K) - e(K) + f(K) = p(v(K') - n) + n - p \cdot e(K') + p \cdot f(K') = p \cdot \chi(S') - pn + n.$$

[7 marks, similar to question set]

(c) Assuming $S \cong S/\!\sim$ implies $\chi(S) = \chi(S/\!\sim)$. Assume that f has n fixed points. Then by (b) one obtains

$$p \cdot \chi(S') - n(p-1) = \chi(S) = \chi(S').$$

Hence, $\chi(S')(p-1) = n(p-1)$ and $\chi(S') = n \geqslant 0$. Hence, by Classification Theorem the only possible surfaces are the sphere, the projective plane, the torus and the Klein bottle, for n being 2, 1 and 0, respectively.

Note, on the other hand, that we have seen that for p an odd prime there exist p-symmetries of S^2 with two fixed points, of $\mathbb{P}^1$ with one fixed point. For the torus we have the rotation by

$2\pi/p$ in one of the factors as a free p-symmetry giving again the torus as a quotient. For the Klein Bottle one should refer to the p-symmetry constructed in the problems, which, as it was shown, gives the Klein bottle again as the quotient. [6 mark, new]

[Total: 17 marks]

Feedback: In (b) many noticed the similarity to one of the example questions and did well with generalising the argument. It was more difficult then expected to see that $\chi(S) = \chi(S/\sim)$ is the starting point for solving the problem.

C10.

(a) Show that if a graph G with v vertices and e edges embeds in a surface of Euler characteristic χ then $\chi \leqslant v - e/3$.

(b) Show that in the situation above for a complete graph G the inequality $v^2 - 7v + 6\chi \leqslant 0$ holds.

(c) Prove that if a complete graph with v vertices embeds into a surface of Euler characteristic $\chi = \frac{7v-v^2}{6}$ then this embedding is 2-cell and every region is bounded by exactly three edges.

(d) Show that the projective plane has a *unique* minimal triangulation.

[17 marks]

Solution

(a) Given an embedding of a graph G with v vertices and e edges in a closed surface S with r regions $2e \geqslant 3r$ holds. Hence, from the inequality $\chi(S) \leqslant v - e + r \leqslant v - e + 2e/3 = v - e/3$.

[2 marks, question set]

(b) The complete graph on v vertices has $\frac{v(v-1)}{2}$ edges. By (a) we have

$$\chi \leqslant v - e/3 = v - \frac{v(v-1)}{6} = \frac{7v - v^2}{6}.$$

Which implies $v^2 - 7v + 6\chi \leqslant 0$. [3 marks, question set]

(c) From the above proof of (b) one sees that $\chi = \frac{7v-v^2}{6}$ holds if and only if $\chi = v - e/3$. Now, the proof of (a) shows that we must have

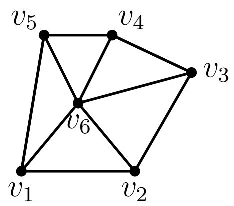
(i) $\chi = v - e + r$, which is the case if and only if the embedding is 2-cell, and

(ii) $3r = 2e$ which implies that every region is bounded by exactly three edges.

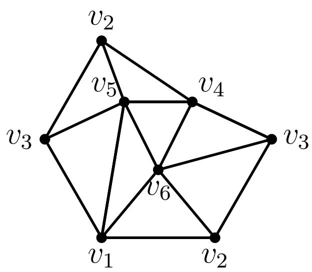
[3 marks, question set]

(d) Assume we have a triangulation of $\mathbb{P}^2$ such that the 1-skeleton is a complete graph on six vertices. Then this triangulation would be minimal. Indeed, by a theorem from the notes every triangulation fulfils $v \geqslant (7 + \sqrt{49 - 24\chi})/2$. Here, we have $6 = (7 + \sqrt{49 - 24})/2$. Hence, it is clearly minimal in terms of vertices. On the other hand, for every triangulation we have $3f = 2e$. It follows that $\chi = v - f/2$ and hence $f = 2(v - \chi)$. Therefore, the triangulation is minimal also with respect to the number of triangles. On the other hand given another minimal triangulation we also must have $v = 6$ and $1 = \chi = 6 - e/3$. Hence, $e = 15 = 6(6-1)/2$ and the 1-skeleton is the complete graph. [3 marks, question set]

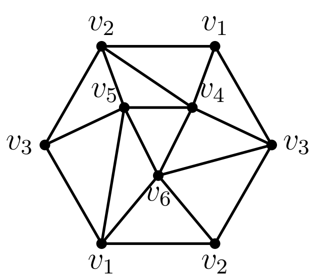
Assume we have a triangulation of $\mathbb{P}^2$ such that the 1-skeleton is a complete graph on six vertices and the vertices are labelled $v_1, \dots, v_6$. Now, the link of the vertex v_6 has to contain all the other vertices. Up to relabelling we may assume that the vertices occur in the cyclic order $v_1, \dots, v_5$ in the link. We obtain the following triangles as a subset of the simplicial surface.



Now, consider the link of v_5. We see that v_1, v_6, v_4 have to occur consecutively in this order. The question is in which order v_2 and v_3 follow in the link. But v_3 cannot follow after v_4, since then the link of v_4 would consist only of v_3, v_5, v_6, which is a contradiction to the fact that the 1-skeleton is a complete graph. Hence, the link of v_5 has to be v_1, v_6, v_4, v_2, v_3. We arrive at the following picture.



Now, consider the link of v_4. We have already v_2, v_5, v_6, v_3 occurring consecutively in this order. Hence, v_1 has to occur between v_3 and v_2 and we arrive at



This gives indeed a triangulation. To see this it remains to check the link condition for v_1, v_2 and v_3. Since in the construction we had no choices this gives the unique triangulation having K_6 as its 1-skeleton. [6 marks, question set]

[Total: 17 marks]

Feedback: Although the problems where all contained on the exercise sheets this question caused the most problems. In particular, (d) was attempted only be a few people. The most common mistake in (d) was to conclude the uniqueness from the fact that the numbers f, e and v are uniquely determined, but in general there might be different triangulations with the same number of vertices/edges and triangles.

C11.

(a) Define what is meant by saying that (X, A) is a *triangulable pair* of spaces.

(b) State the axioms for the *reduced homology groups* of triangulable spaces.

(c) Prove that if there is a short exact sequence of abelian groups

$$0 \to H \xrightarrow{i} G \xrightarrow{q} \mathbb{Z} \to 0$$

then $G \cong H \times \mathbb{Z}$.

(d) Determine the reduced homology groups of the disjoint union of a triangulable space X and a single point in terms of the groups $\tilde{H}_i(X)$, $i \geqslant 0$.

[16 marks]

Solution

(a) A *triangulable pair* of spaces (X, A) is a topological space X with a subspace A such that there is a homeomorphism $h\colon X \to |K|$, the underlying space of a simplicial complex K, with $h(A) = |L|$ the underlying space of a subcomplex L of K. [2 mark, bookwork]

(b) A *reduced homology theory* assigns to each non-empty triangulable space X a sequence of abelian groups $\tilde{H}_n(X)$ for $n \in \mathbb{Z}$ and for each continuous map of triangulable spaces $f\colon X \to Y$ a sequence of homomorphisms $f_*\colon \tilde{H}_n(X) \to \tilde{H}_n(Y)$ such that the following axioms hold.

(i) [Functorial Axiom 1] Given continuous functions $f\colon X \to Y$ and $g\colon Y \to Z$, it follows that

$$g_* \circ f_* = (g \circ f)_*\colon \tilde{H}_n(X) \to \tilde{H}_n(Z) \text{ for all } i.$$

(ii) [Functorial Axiom 2] For the identity map $\mathrm{id}_X\colon X \to X$,

$$(\mathrm{id}_X)_* = \mathrm{id}_{H_n(X)}\colon \tilde{H}_n(X) \to \tilde{H}_n(X) \text{ (the identity homomorphism) for all } n.$$

(iii) [Homotopy Axiom] For homotopic maps $f \simeq g\colon X \to Y$,

$$f_* = g_*\colon \tilde{H}_n(X) \to \tilde{H}_n(Y) \text{ for all } n.$$

(iv) [Exactness Axiom] For any triangulable pair (X, A) there are boundary homomorphisms $\partial\colon \tilde{H}_n(X/A) \to \tilde{H}_{n-1}(A)$ for all n which fit into a long exact sequence as follows.

$$\ldots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{q_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \xrightarrow{i_*} \tilde{H}_{n-1}(X) \to \ldots$$

Furthermore, given any continuous function of triangulable pairs $f\colon (X, A) \to (Y, B)$ (i.e. $f\colon X \to Y$ such that $f(A) \subset B$) this induces a continuous function of quotient spaces $\bar{f}\colon X/A \to Y/B$. Then the following diagram commutes for all n.

$$\begin{tikzcd} \tilde{H}_n(X/A) \arrow[r, "\partial"] \arrow[d, "\bar{f}_*"] & \tilde{H}_{n-1}(A) \arrow[d, "f_*"] \\ \tilde{H}_n(Y/B) \arrow[r, "\partial"] & \tilde{H}_{n-1}(B) \end{tikzcd}$$

(v) [Dimension Axiom] $\tilde{H}_0(S^0) \cong \mathbb{Z}$ and $\tilde{H}_n(S^0) = 0$ for all $n \neq 0$.

[7 marks, bookwork]

(c) Since q is surjective, we may choose an element $g_1 \in G$ such that $q(g_1) = 1$. Then we may define a homomorphism $f\colon H \times \mathbb{Z} \to G$ by $f(h, n) = i(h) + ng_1$. This is clearly a homomorphism since i is a homomorphism. To see that f is surjective suppose that $g \in G$ and $q(g) = n \in \mathbb{Z}$. Then $q(g - ng_1) = q(g) - nq(g_1) = n - n = 0$ so that $g - ng_1 \in \mathrm{Ker}(q) = \mathrm{Im}(i)$ (by exactness) and so $g - ng_1 = i(h)$ for some $h \in H$ so that $g = i(h) + ng_1 = f(h, n)$ as required. To see that f is injective suppose that $f(h, n) = i(h) + ng_1 = 0$. Then $q(i(h) + ng_1) = 0$ gives $n = 0$. Hence, $i(h) = 0$ so that $h = 0$ since i is injective. Hence, f is an isomorphism.

[2 marks, problem set]

(d) Consider the pair $(X \sqcup *, X)$ for which $(X \sqcup *)/X = * \sqcup * = S^0$. Then the exactness axiom gives

$$\ldots \to \tilde{H}_{n+1}(S^0) \xrightarrow{\partial} \tilde{H}_n(X) \xrightarrow{i_*} \tilde{H}_n(X \sqcup *) \xrightarrow{q_*} \tilde{H}_n(S^0) \to \ldots$$

For $n > 0$ we have $\tilde{H}_{n+1}(S^0) = \tilde{H}_n(S^0) = 0$ by Dimension Axiom. Hence, $\tilde{H}_n(X) \cong \tilde{H}_n(X \sqcup *)$. For $n = 0$ one obtains

$$\ldots \to 0 \xrightarrow{\partial} \tilde{H}_0(X) \xrightarrow{i_*} \tilde{H}_0(X \sqcup *) \xrightarrow{q_*} \mathbb{Z} \to 0$$

Hence, by (b) one obtains $\tilde{H}_0(X \sqcup *) = \tilde{H}_0(X) \times \mathbb{Z}$. [5 marks, similar to problem set]

[Total: 16 marks]

Feedback: When attempted parts (a) and (b) were done well. Part (c) was meant mainly to help with (d). However, it caused some problems. There were only very few correct solutions for (d). Most people didn't recognise the necessity of using the exactness axiom. One mistake was to choose the wrong subspace for the pair (the single point instead of X, which doesn't help since $Y/*$ is always Y by definition).

END OF EXAMINATION PAPER