§1. Topological Surfaces

1.1 Definition. Let n be a non-negative integer. An *n-dimensional (topological) manifold* (or a *(topological) n-manifold* is a topological space X which

(i) is Hausdorff,

(ii) is second countable (i.e. has a countable basis), and

(iii) is *locally Euclidean*, i.e. each point $x \in X$ lies in an open subset V in X which is homeomorphic to an open subset $U \subset \mathbb{R}^n$ (with the usual topology).

A homeomorphism $\phi\colon U \to V$ where U is an open subset of $\mathbb{R}^n$ and V is an open subset of X is called a *chart* or a *parametrisation* of the open set V. The inverse $\phi^{-1}\colon V \to U$ is called a *coordinate system* on V.

A collection of charts $\{\, \phi_\lambda\colon U_\lambda \to V_\lambda \mid \lambda \in \Lambda \,\}$ is called an *atlas* on X when each point of X lies in some open set V_λ.

A topological 1-manifold is called a *topological curve.*

A topological 2-manifold is called a *topological surface.*

1.2 Remarks. The Hausdorff condition ensures that there are enough open sets and the second countable condition ensures that there are not too many. These conditions are required for technical reasons. There are some examples on Problems 1 of locally Euclidean spaces which do not satisfy these conditions.

1.3 Examples. (a) Any open subset V of $\mathbb{R}^n$ (including $\mathbb{R}^n$ itself) is a topological n-manifold with a single chart $\mathrm{id}_V\colon V \to V$.

(b) The 2-sphere $S^2 = \{\, x \in \mathbb{R}^3 \mid |x| = 1 \,\}$ is a topological surface.

To see this let $V_3^+ = \{\, \mathbf{x} = (x_1, x_2, x_3) \in S^2 \mid x_3 > 0 \,\}$ (the open northern hemisphere). This set may be parametrised by the chart $\phi_3^+\colon B_1(\mathbf{0}) = \{\, \mathbf{u} \in \mathbb{R}^2 \mid |\mathbf{u}| < 1 \,\} \to V_3^+$ defined by

$$\phi_3^+(u_1, u_2) = (u_1, u_2, +\sqrt{1 - |\mathbf{u}|^2}).$$

This is continuous because the component functions are continuous and it is a homeomorphism since it has continuous inverse given by $(u_1, u_2, u_3) \mapsto (u_1, u_2)$. So it gives a chart on S^2 around points in the open northern hemisphere.

Similarly we may define a chart $\phi_3^-\colon B_1(\mathbf{0}) \to V_3^- = \{\, \mathbf{x} \in S^2 \mid x_3 < 0 \,\}$ given by $\mathbf{u} \mapsto (\mathbf{u}, -\sqrt{1 - |\mathbf{u}|^2})$ around points in the open southern hemisphere.

In the same way we may define two charts

$$\phi_1^\pm\colon B_1(\mathbf{0}) \to \{\, \mathbf{x} \in S^2 \mid \pm x_1 > 0 \,\}$$

by $\mathbf{u} \mapsto (\pm\sqrt{1 - |\mathbf{u}|^2}, \mathbf{u})$ and two charts

$$\phi_2^\pm\colon B_1(\mathbf{0}) \to \{\, \mathbf{x} \in S^2 \mid \pm x_2 > 0 \,\}$$

by $\mathbf{u} \mapsto (u_1, \pm\sqrt{1 - |\mathbf{u}|^2}, u_2)$.

These six charts provide an atlas for S^2 because each point of S^2 has at least one non-zero coordinate and so lies in at least one of the open subsets $V_i^\pm$.

(c) More generally, $S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid |x| = 1 \,\}$ is a topological n-manifold and has an atlas with $2(n+1)$ charts generalizing the method of the previous example.

1.4 Proposition. If M_1 is a topological n_1-manifold and M_2 is a topological n_1-manifold then the product space $M_1 \times M_2$ is a topological $(n_1 + n_2)$-manifold.

Proof. Use products of charts (exercise). □

1.5 Examples. The cylinder $S^1 \times (0, 1)$ and the torus $S^1 \times S^1$ are topological surfaces.

The projective plane

1.6 Definition. Define an equivalence relation on S^2 by $\mathbf{x} \sim \pm\mathbf{x}$ for all $\mathbf{x} \in S^2$. Then each equivalence class $[\mathbf{x}] = \{\mathbf{x}, -\mathbf{x}\}$ is a pair of antipodal points on S^2. We define the *projective plane* P^2 to be the set of equivalence classes $S^2/{\sim}$ with the quotient topology.

1.7 Proposition. The projective plane P^2 is a topological surface.

Proof. We can provide an atlas of three charts based on the atlas for S^2 given above as follows.

Let $\phi_3\colon B_1(\mathbf{0}) \to P^2$ be given by $\phi_3(\mathbf{u}) = [\phi_3^+(\mathbf{u})] = [\mathbf{u}, \sqrt{1 - |\mathbf{u}|^2}]$ (using the notation of Example 1.3(b)). Then ϕ_3 gives a continuous bijection $B_1(\mathbf{0}) \to V_3 = \{\, [\mathbf{x}] \in P^2 \mid x_3 \neq 0 \,\}$ since, given $[\mathbf{x}] \in V_3$, then $[(x_1, x_2, x_3)] = [(-x_1, -x_2, -x_3)]$ and precisely one of x_3 and $-x_3$ is positive. V_3 is open in P^2 since $q^{-1}(V_3) = V_3^+ \cup V_3^- \subset S^2$. ϕ_3^{-1} is continuous since, if U is an open subset of $B_1(\mathbf{0})$, then $\phi_3(U)$ is open in P^2 (and so in V_3 since V_3 is open in P^2) since $q^{-1}\phi_3(U) = \phi_3^+(U) \cup \phi_3^-(-U)$ which is open in S^2. Hence $\phi_3\colon B_1^2(\mathbf{0}) \to V_3$ is a chart on P^2.

Similarly, we may define charts

$$\phi_1\colon B_1(\mathbf{0}) \to V_1 = \{\, [\mathbf{x}] \in P^2 \mid x_1 \neq 0 \,\}, \quad \phi_1(\mathbf{u}) = [\phi_1^+(\mathbf{u})] = [(\sqrt{1-|\mathbf{u}|^2}, \mathbf{u})],$$

$$\phi_2\colon B_1(\mathbf{0}) \to V_1 = \{\, [\mathbf{x}] \in P^2 \mid x_2 \neq 0 \,\}, \quad \phi_1(\mathbf{u}) = [\phi_2^+(\mathbf{u})] = [(u_1, \sqrt{1-|\mathbf{u}|^2}, u_2)].$$

These three charts cover P^2 since $V_1 \cup V_2 \cup V_3 = P^2$. Hence P^2 is locally Euclidean.

To see that P^2 is Hausdorff suppose that $[\mathbf{x}]$, $[\mathbf{y}]$ are two distinct points of $\mathbb{R}^2$. Let $\varepsilon = \frac{1}{2} \cdot \min(|\mathbf{x}-\mathbf{y}|, |\mathbf{x}+\mathbf{y}|)$. Then the four open sets $B_\varepsilon^{S^2}(\mathbf{x})$, $B_\varepsilon^{S^2}(-\mathbf{x})$, $B_\varepsilon^{S^2}(\mathbf{y})$, $B_\varepsilon^{S^2}(-\mathbf{y})$ are disjoint and so $q\big(B_\varepsilon^{S^2}(\mathbf{x})\big)$ and $q\big(B_\varepsilon^{S^2}(\mathbf{y})\big)$ are the required disjoint open sets in P^2 containing $[\mathbf{x}]$ and $[\mathbf{y}]$.

Finally P^2 is second countable by the following result. $\square$

1.8 Proposition. A compact space which is locally Euclidean is second countable.

Proof. Exercise. Observe that a compact locally Euclidean space has a finite atlas. $\square$

1.9 Remark. In the same say we may define projective n-space $P^n = S^n/(\mathbf{x} \sim \pm\mathbf{x})$ and this is a topological n-manifold.

The connected sum of two surfaces

1.10 Remark. The surfaces S^2, $S^1 \times S^1$ and P^2 are basic in the sense that any non-empty compact surface may be obtained from them by a process known as *connected sum.* Roughly speaking, the connected sum of two surfaces S_1 and S_2 is formed by removing open discs from each and then gluing them together along the boundary circles of the resulting holes. We first need to observe that we can find such discs.

1.11 Proposition. Given a topological surface S it has an atlas consisting of charts of the form $\phi\colon B_1(\mathbf{0}) \to V \subset S$.

Proof. Exercise. $\square$

1.12 Definition. Suppose that S_1 and S_2 are non-empty path-connected topological surfaces. Choose subspaces $V_1 \subset S_1$ and $V_2 \subset S_2$ which are homeomorphic to the open disc $B_1(\mathbf{0}) \subset \mathbb{R}^2$ by homeomorphisms

$$\phi_i \colon B_1(\mathbf{0}) \to V_i \quad \text{for } i = 1 \text{ and } i = 2$$

(which we can do by Proposition 1.11).

We remove the interiors of smaller discs, i.e. $\phi_i\big(B^2_{1/2}(\mathbf{0})\big)$ and glue along the boundary circles. More precisely, we define the quotient space of the disjoint union

$$S = \Big[\Big(S_1 - \phi_1\big(B^2_{1/2}(\mathbf{0})\big)\Big) \sqcup \Big(S_2 - \phi_2\big(B^2_{1/2}(\mathbf{0})\big)\Big)\Big] \Big/{\sim}$$

where $\phi_1(\mathbf{u}) \sim \phi_2(\mathbf{u})$ for $\mathbf{u} \in B^2_1(\mathbf{0})$ with $|\mathbf{u}| = 1/2$.

We call S the *connected sum* of S_1 and S_2 and write this as $S_1 \# S_2$.

1.13 Theorem. The connected sum $S_1 \# S_2$ of two path-connected surfaces S_1 and S_2 is well-defined up to homeomorphism and does not depend on the choice of charts $\phi_i \colon B_1(\mathbf{0}) \to V_i$.

Proof. This is rather technical and is omitted. For compact surfaces this is in fact a consequence of the classification theorem (Theorem 1.15 below).□

1.14 Theorem. The connected sum $S_1 \# S_2$ of two path-connected topological surfaces, S_1 and S_2, is a path-connected topological surface.

Proof. First of all notice that the open subspace

$$U = \Big[\Big(V_1 - \phi_1\big(B^2_{1/2}(\mathbf{0})\big)\Big) \sqcup \Big(V_2 - \phi_2\big(B^2_{1/2}(\mathbf{0})\big)\Big)\Big] \Big/{\sim}$$

of $S_1 \# S_2$ is homeomorphic to the topological surface $(0,1) \times S^1$ via a homeomorphism given by $f \colon (0,1) \times S^1 \to U$ given by

$$f(t, \mathbf{x}) = \begin{cases} \big[\phi_1\big((1-t)\mathbf{x}\big)\big], & \text{for } 0 < t \leqslant 1/2, \\ [\phi_2(t\mathbf{x})], & \text{for } 1/2 \leqslant t < 1. \end{cases}$$

[This function and its inverse are continuous by the Gluing Lemma.]

Furthermore the two open sets in $S_1 \# S_2$ given by the homeomorphic images of

$$U_1 = S_1 \setminus \phi_1(\{\, \mathbf{u} \in \mathbb{R}^2 \mid |\mathbf{u}| \leqslant 1/2 \,\}),$$
$$U_2 = S_2 \setminus \phi_2(\{\, \mathbf{u} \in \mathbb{R}^2 \mid |\mathbf{u}| \leqslant 1/2 \,\})$$

are also topological surfaces.

Now each point of $S_1 \# S_2$ lies in one or more of these three open subsets and so since they are each topological surfaces it has on open neighbourhood in the open subset and so in $S_1\#S_2$ which is homeomorphic to an open subset of $\mathbb{R}^2$ as required to prove that $S_1\#S_2$ is locally Euclidean.

If S_1 and S_2 are compact then so is $S_1\#S_2$ which is therefore second countable by Proposition 1.8. The proof of second countability in the general case is omitted.

We can see that $S_1\#S_2$ is Hausdorff by considering various possibilities. The details are omitted.

$S_1\#S_2$ is path-connected since there is a path between every point and $[\phi_1(\mathbf{u})] = [\phi_2(\mathbf{u})]$ for any point $u \in \mathbb{R}^2$ with $|\mathbf{u}| = /1/2$. $\square$

The classification theorem for compact surfaces

1.15 Theorem [Classification Theorem for Closed Surfaces]. Every path-connected compact topological surface (or *closed surface*) is homeomorphic to one and only one of:

(a) S^2,

(b) T_g for some $g \geqslant 1$ (where $T_1 = S^1 \times S^1$ and $T_{g+1} = T_g\#T_1$ for $g \geqslant 1$),

(c) P_g for some $g \geqslant 1$ (where $P_1 = P^2$ and $P_{g+1} = P_g\#P_1$ for $g \geqslant 1$).

The number g is called the *genus* of the surface.

The course is structured around the proof of this theorem. §2 is concerned with the proof that every path-connected compact topological surface is homeomorphic to a surface on the list and the remainder of the course deal with the topological invariants needed to show that the surfaces on the list are all topologically distinct.

Cut and paste arguments

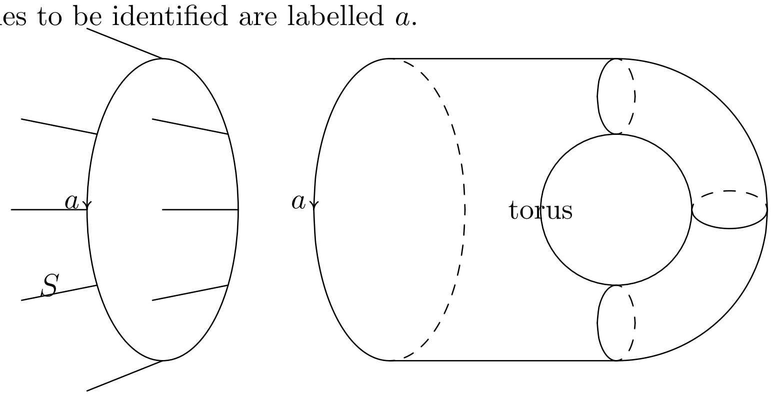
1.16 Remarks. In this course we shall often be rather informal in constructing topological spaces and homeomorphisms. Continuing to do this in full detail would become extremely tedious. So what follows are a number of results illustrating these more informal arguments.

1.17 Proposition. For any path-connected topological surface S, $S\#S^2$ is homeomorphic to S.

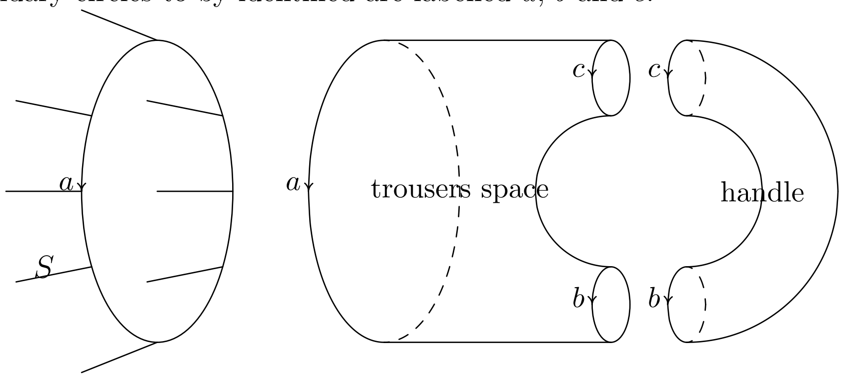
Outline proof. If an open disc is removed from S^2 then what is left is a closed disc. Thus $S\#S^2$ is formed by removing the interior of a closed disc from S and then attaching a closed disc by the boundary circle which takes us back to S. $\square$

1.18 Proposition. For any path-connected topological surface S, $S\#T_1$ can be thought of as S with a *handle*, i.e. we remove the interiors of two closed discs in S and then attach a closed cylinder ($S^1 \times I$) by the two boundary circles.

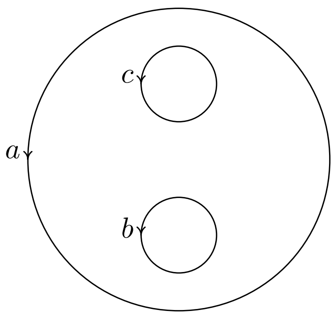
Outline proof. We may picture the connected sum of a surface S with the torus as the following identification space where the two boundary boundary circles to be identified are labelled a.



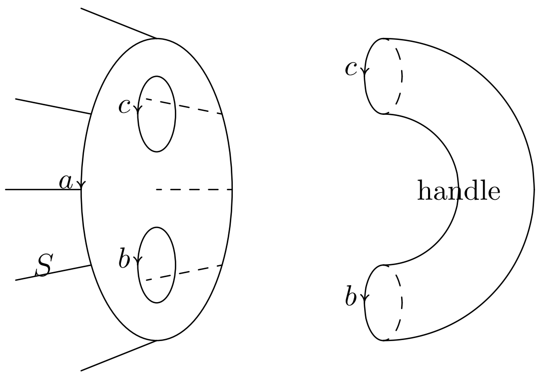
Now the torus may be obtained by gluing together two closed cylinders and so thinking of the torus in this way we can the connected sum of the surface S and the torus and following identification space where the three pairs of boundary circles to by identified are labelled a, b and c.



Here the middle space, which is obtained by removing an open disc, from the closed cylinder is sometimes called 'the trousers space' for fairly obvious reasons (a is the waist and b and c are the cuffs). It is homeomorphic the a closed disc with two open discs removed as follows (the cylinder is homeomorphic to an annulus and this is an annulus with an additional open disc removed).



Now replacing the trousers space by the disc with two holes and then doing the identification a gives rise the following identification space.



This gives S with a handle attached. $\square$

1.19 Definition. Let $I = [0, 1]$ with the usual topology. Then the *Möbius band* is the identification space $I^2/\!\sim$ where $(s, 0) \sim (1 - s, 1)$. The image of the subset $\{0, 1\} \times I$ in the Möbius band is called the *boundary* of the band and is homeomorphic to the circle S^1.

1.20 Proposition. The identification space obtained by gluing together a Möbius band and a closed disc by their boundary circles is homeomorphic to the projective plane.

Proof. Exercise. Recall that the projective plane is homeomorphic the identification space $D^2/\!\sim$ where $\mathbf{x} \sim \mathbf{x}' \in D^2$ if and only if $\mathbf{x} = \mathbf{x}'$ or $\mathbf{x}' = -\mathbf{x} \in S^1 \subset D^2$ (see MATH31051/41051/61051 Problems 4, Question 8). $\square$

1.21 Proposition. For any path-connected topological surface S, $S \# P_1$ can be thought of S with a *cross-cap*, i.e. we remove the interior of a closed disc in S and then attach a Möbius band by the boundary circles.

Outline proof. This follows by a similar argument to Proposition 1.18 using Proposition 1.20. $\square$

1.22 Remarks. (a) It is a consequence of Propositions 1.18 and 1.21 that the Classification Theorem 1.15 may be interpreted as saying that every path-connected topological surface is homeomorphic to a sphere with g handles attached or a sphere with g cross-caps.

(b) It is not clear from this what surface is obtained if both a handle and a cross-cap are attached to a sphere. An equivalent question is to ask where the surface $T_1 \# P_1$ is in the Classification Theorem.