MATH41071/MATH61071 *Algebraic topology*

5 Simplicial Maps, Simplicial Approximations and the Invariance of Homology

Simplicial maps

5.1 Definition. Let K and L be simplicial complexes. Then an *admissible vertex map* from K to L is a function $\phi\colon V(K) \to V(L)$ such that, if $\{ v_0, v_1, \dots, v_r \}$ is a simplex of K, then $\{ \phi(v_0), \phi(v_1), \dots, \phi(v_r) \}$ is a simplex of L.

Note. ϕ need not be an injection and so the simplex $\{ \phi(v_0), \phi(v_1), \dots, \phi(v_r) \}$ may contain repeated vertices in which case it is a simplex of dimension less than r.

5.2 Proposition. An admissible vertex map ϕ from K to L induces a continuous map of the underlying spaces

$$|\phi|\colon |K| \to |L|$$

by linear extension of the vertex map

$$|\phi|\Big(\sum_{i=0}^{r} t_i v_i\Big) = \sum_{i=0}^{r} t_i \phi(v_i).$$

Proof. The function $|\phi|$ is well-defined and continuous by the Gluing Lemma since the simplices are closed subsets of $|K|$. $\square$

5.3 Definition. A continuous function $f\colon |K| \to |L|$ between the underlying spaces of two simplicial complexes is a *simplicial map* (with respect to K and L) if $f = |\phi|$ for an admissible vertex map from K to L.

5.4 Proposition. Suppose that $\phi\colon V(K) \to V(L)$ is an admissible vertex map from a simplicial complex K to a simplicial complex L. Then ϕ induces homomorphisms

$$\phi_*\colon C_r(K) \to C_r(L)$$

defined on generators by

$$\phi_*(\langle v_0, v_1, \dots, v_r \rangle) = \begin{cases} \langle \phi(v_0), \phi(v_1), \dots, \phi(v_r) \rangle & \text{if the vertices } \phi(v_i) \\ & \text{are all distinct,} \\ 0 & \text{otherwise.} \end{cases}$$

The homomorphisms ϕ_* commute with the boundary homomorphisms as follows.

$$\begin{tikzcd} C_r(K) \arrow[r, "\phi_*"] \arrow[d, "d_r"] & C_r(L) \arrow[d, "d_r"] \\ C_{r-1}(K) \arrow[r, "\phi_*"] & C_{r-1}(L) \end{tikzcd} \tag{1}$$

Hence, ϕ_* induces homomorphisms $\phi_*\colon Z_r(K) \to Z_r(L)$ and $\phi_*\colon B_r(K) \to B_r(L)$ and so homomorphisms

$$\phi_*\colon H_r(K) \to H_r(L).$$

Proof. To check the commutativity of the diagram we evaluate $d_r \circ \phi_*$ and $\phi_* \circ d_r$ on the generators. On an r-simplex who vertices are mapped injectively, the check is a routine calculation. In other cases ϕ_* maps the simplex to 0. If the $(r+1)$ vertices of an r-simplex are mapped to fewer than r vertices this means all of the faces of the simplex are also mapped to 0 and so commutativity is immediate. If the $(r+1)$ vertices of an r-simplex in K are mapped to r vertices, order the vertices so that $\phi(v_0) = \phi(v_1)$. Then all the $(r-1)$-faces of the simplex are mapped by ϕ_* to 0 except that $\langle v_0, v_2, \dots, v_r\rangle \mapsto \langle \phi(v_0), \phi(v_2), \dots, \phi(v_r)\rangle$ and $\langle v_1, v_2, \dots, v_r\rangle \mapsto \langle \phi(v_1), \phi(v_2), \dots, \phi(v_r)\rangle$. When $\phi_* \circ d_r$ is evaluated on this r-simplex these two terms cancel out giving 0, the same result as evaluating $d_r \circ \phi_*$.
To see that ϕ_* induces maps as described:

$$x \in Z_r(K) \Rightarrow d_r(x) = 0 \Rightarrow d_r\phi_*(x) = \phi_* d_r(x) = \phi_*(0) = 0 \Rightarrow d_r(x) \in Z_r(L);$$

$$x \in B_r(K) \Rightarrow x = d_{r+1}(x') \Rightarrow \phi_*(x) = \phi_* d_{r+1}(x') = d_{r+1}\phi_*(x') \Rightarrow \phi_*(x) \in B_r(L).$$

Now we can define $\phi_*\colon H_r(K) = Z_r(K)/B_r(L) \Rightarrow H_r(L) = Z_r(L)/B_r(L)$ by

$$\phi_*(B_r(K) + x) = B_r(L) + \phi_*(x).$$

□

5.5 Definition. A family of homomorphism $\phi_* : C_r(K) \to C_r(L)$ as in the proposition, i.e. making the diagrams in (2) commutative, is called a *chain map*. The proof shows that even more general chain maps induce homomorphisms of homology groups.

5.6 Remark. Thus a simplicial map $|K| \to |L|$ between the underlying spaces of two simplicial complexes induces a homomorphism of the homology groups. The difficulty with this is that most continuous maps $|K| \to |L|$ are not simplicial maps and there are very few simplicial maps, certainly only finitely many, whereas the number of continuous maps is uncountable in most cases. In order to permit the construction of more simplicial maps we subdivide the simplices along the lines of Definition 3.13. This still will not enable us to make all continuous maps simplicial but it will allow us to approximate all continuous functions $|K| \to |L|$ by simplicial maps. We first look at the approximation method.

Homotopic maps and homotopy equivalent spaces

5.7 Definition. Two continuous functions of topological spaces $f_0 \colon X \to Y$ and $f_1 \colon X \to Y$ are *homotopic*, written $f_0 \simeq f_1$, if there is a continuous map $H \colon X \times I \to Y$ such that $H(x,0) = f_0(x)$ and $H(x,1) = f_1(x)$. We call H a *homotopy* between f_0 and f_1 and write $H \colon f_0 \simeq f_1 \colon X \to Y$.

5.8 Remark. (a) If $H \colon f_0 \simeq f_1 \colon X \to Y$ then there is a 'continuous family' of continous functions $f_t \colon X \to Y$ between f_0 and f_1 given by $f_t(x) = H(x,t)$.

(b) If A is a subspace of X and f_0, $f_1 \colon X \to Y$ are two functions which agree on A: $f_0(a) = f_1(a)$ for $a \in A$ then we say that f_0 and f_1 are *homotopic relative to* A, written $f_0 \simeq f_1$ rel A, if there is a homotopy $H \colon f_0 \simeq f_1$ such that $H(a,t) = f_0(a)$ for all $a \in A$.
A particular case of this is when $X = I$ and $A = \{0,1\}$. Then continuous functions $\sigma_0, \sigma_1 \colon I \to Y$ such that $\sigma_(0) = \sigma_1(0) = y_0$ and $\sigma_0(1) = \sigma_1(1) = y_1$ (i.e. two paths in Y from y_0 to y_1) are homotopic relative to $\{0,1\}$ when they are equivalent paths.

5.9 Proposition. Homotopy is an equivalence relation on the set of all continuous functions from a topological space X to a topological space Y:

(i) $f \simeq f \colon X \to Y$;

(ii) $f_0 \simeq f_1 \colon X \to Y \Rightarrow f_1 \simeq f_0 \colon X \to Y$;

(iii) $f_0 \simeq f_1 \colon X \to Y$ and $f_1 \simeq f_2 \colon X \to Y \Rightarrow f_0 \simeq f_2 \colon X \to Y$.

Proof. Exercise. □

5.10 Definition. A continuous function $f \colon X \to Y$ is a *homotopy equivalence* when there it has a *homotopy inverse* $g \colon Y \to X$ which means that $g \circ f \simeq I \colon X \to X$, the identity map, and $f \circ g \simeq I \colon Y \to Y$. In this case we say that X and Y are *homotopy equivalent* spaces and denote this by $X \equiv Y$ (or sometimes $X \simeq Y$).

Notice that a homeomorphism is trivially a homotopy equivalence.

5.11 Definition. If a topological space X is homotopy equivalent to a one-point space then it is said to be *contractible.*

5.12 Definition. A subspace A of a topological space X is said to be a *deformation retract* when there is a *retract* $r \colon X \to A$ (i.e. a continuous function such that $r(a) = a$ for all $a \in A$) such that $i \circ r \simeq I \colon X \to X$.

This is the most common way of constructing a homotopy equivalence.

5.13 Example. (a) $\mathbb{R}^n$ is contractible since if we consider the inclusion map $i\colon \{\mathbf{0}\} \to \mathbb{R}^n$ and the constant map $c\colon \mathbb{R}^n \to \{\mathbf{0}\}$ then $c \circ i = I\colon \{\mathbf{0}\} \to \{\mathbf{0}\}$ and a homotopy $H\colon i \circ c \simeq I\colon \mathbb{R}^n \to \mathbb{R}^n$ is defined by $H(\mathbf{x}, t) = t\mathbf{x}$. So $\{\mathbf{0}\}$ is a deformation retract of $\mathbb{R}^n$.

(b) The $(n-1)$-sphere S^{n-1} is homotopy equivalent to the punctured ball $D^n \backslash \{\mathbf{0}\}$. For let $i\colon S^{n-1} \to D^n \backslash \{\mathbf{0}\}$ be the inclusion map and $r\colon D^n \backslash \{\mathbf{0}\} \to S^{n-1}$ the radial projection map $r(\mathbf{x}) = \mathbf{x}/|\mathbf{x}|$. Then $r \circ i = I\colon S^{n-1} \to S^{n-1}$ and a homotopy $H\colon i \circ r \simeq I\colon D^n \setminus \{\mathbf{0}\} \to D^n \setminus \{\mathbf{0}\}$ is given by $H(\mathbf{x}, t) = (1-t)\mathbf{x}/|\mathbf{x}| + t\mathbf{x}$. So S^{n-1} is a deformation retract of $D^n \setminus \{\mathbf{0}\}$.

5.14 Proposition. (a) If a topological space X is contractible then it is path-connected.

(b) If a topological space is X is contractible then, for every point $x_0 \in X$, $\{x_0\}$ is a deformation retract of X.

Proof. Exercise. □

Barycentric subdivisions and the simplicial approximation theorem

5.15 Definition. Suppose that $f\colon |K| \to |L|$ is a continuous function between the underlying spaces of two simplicial complexes K and L.
Notice that, for each point $y \in |L|$, y is an interior point of a unique simplex $\sigma \in L$. This simplex is called the *carrier* of y and is denoted $\mathrm{carr}_L(y)$ (with respect to L)
A simplicial map $|\phi|\colon |K| \to |L|$ is a *simplicial approximation* to f (with respect to K and L) if, for each point $x \in |K|$ the image $|\phi|(x)$ lies in the carrier of $f(x)$.

5.16 Remark. One can check that the composition of simplicial approximations of $f : |K| \to |L|$ and $g : |L| \to |M|$, respectively, gives a simplicial approximation of $g \circ f$.

5.17 Proposition. If the simplicial map $|\phi|\colon |K| \to |L|$ is a simplicial approximation to the continuous function $f\colon |K| \to |Y|$ then $|\phi| \simeq f\colon |K| \to |L|$.

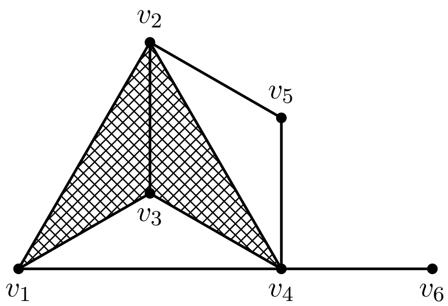
Proof. Define $H\colon |K| \times I \to |L|$ by $H(x,t) = (1-t)|\phi|(x) + tf(x) \in |L|$. This definition makes sense since both of $|\phi|(x)$ and $f(x)$ lie in the carrier of $f(x)$ which is a convex subset of Euclidean space. The function is clearly continuous by the algebra of continuous functions. □

5.18 Example. Not every function $f : |K| \to |L|$ has a simplicial approximation with respect to a given K and L. Consider the continuous function

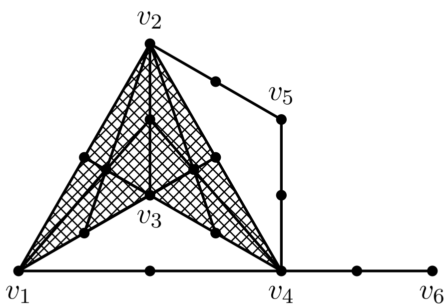
$f\colon [0,1] \to [0,1]$ given by $f(x) = x^2$. Let K be the simplicial complex in $\mathbb{R}$ with simplices $\langle 0, 1/2\rangle$ and $\langle 1/2, 1\rangle$ and their faces so that $|K| = [0,1]$. Then f does not have a simplicial approximation $|\phi|\colon |K| \to |K|$.
To prove this suppose that $\phi\colon V(K) \to V(K)$ is an admissible vertex map so that $|\phi|$ is a simplicial approximation to f. Since $f(0) = 0$, the carrier of $f(0)$ is the vertex $\langle 0\rangle$ and so $|\phi(0)| \in \langle 0\rangle$, i.e. $\phi(0) = 0$.
Similarly, $\phi(1) = 1$ since $f(1) = 1$.
This means, for ϕ to be admissible we must have $\phi(1/2) = 1/2$ (since, for example, if $\phi(1/2) = 0$ then $\langle 1/2, 1\rangle$ is a simplex in K but $\langle \phi(1/2), \phi(1)\rangle = \langle 0, 1\rangle$ is not a simplex in K, and similarly, $\phi(1/2) = 1$ is not possible). Thus $|\phi| = I\colon |K| \to |K|$, the identity map,
But the identity map is not a simplicial approximation to f since, for example, if $x = 2/3$, then $f(2/3) = 4/9$ which has the carrier $\langle 0, 1/2\rangle$ whereas $|\phi|(2/3) = 2/3$ does not lie in this simplex.

5.19 Definition. Recall from Proposition 4.9 that the coefficients $(t_0, t_1, \ldots, t_k)$ of a point $x = \sum_{i=0}^{k} t_i v_i \in \langle v_0, v_1, \ldots, v_k\rangle$ are called the barycentric coordinates of x.
The *barycentre* $\hat{\sigma}$ of a k-simplex σ $(k \geqslant 0)$ is the point $(\sum_{i=0}^{k} v_i)/(k+1)$, the 'centre of mass' of the vertices with barycentric coordinates $\big(1/(k+1), 1/(k+1), \ldots, 1/(k+1)\big)$.
Given a simplicial complex K, the *first barycentric subdivision of* K, denoted K', is a simplicial complex. The vertices of K' are the barycentres of the (non-empty) simplices of K. The k-simplices of K' are given by $\langle \hat{\sigma}_0, \hat{\sigma}_1, \ldots \hat{\sigma}_k\rangle$ where, for each i, $1 \leqslant i \leqslant k$, σ_{i-1} is a proper face of σ_i in K.

5.20 Example. Suppose that K is the simplicial complex in $\mathbb{R}^2$ which could be pictured as follows.



The first barycentric subdivision is as follows.



5.21 Proposition. Given a simplicial complex K, its first barycentric subdivision K' is also a simplicial complex of the same dimension as K with the following properties.

(a) Each simplex of K' is a subset of a simplex of K.

(b) $|K'| = |K|$.

5.22 Definition. The mth barycentric subdivision $K^{(m)}$ of a simplicial complex K is defined inductively by $K^{(i+1)} = (K^{(i)})'$ for $i \geqslant 1$.

The barycentric subdivision step can be further decomposed into simpler operations, namely generalisations of the starring operations, which we have seen before.

5.23 Definition. Choose a point v in the interior of some simplex $\tau = \langle v_0, \ldots, v_r \rangle$ of a simplicial complex K. The *starring* of K in v is defined to be the simplicial complex $K(v)$, whose set of simplices consists of

(a) simplices of K which do not have τ as a face

(b) simplices of the form $\langle v, w_0, \ldots, w_s \rangle$, where $\{w_0, \ldots, w_s\}$ is a subset of vertices of a simplex σ with $\tau \prec \sigma$ and at least on vertex of τ is not contained in $\{w_0, \ldots, w_s\}$ (i.e. in particular $s < r$).

Now, the barycentric subdivision is obtained by, first starring in the barycentres of n-simplices of K. The result will contain the original $(n-1)$-simplices of K. In the second step we are starring in the barycentres of these and the result will still contain the $(n-2)$-simplices of K, in whose barycentres we are starring next, and so on. After starring in the barycentres of 1-simplices (edges) the result is the barycentric subdivision.

5.24 Example (Example 5.18 continued). We might now consider whether the function $f(x) = x^2$ has a simplicial approximation $|\phi|\colon |K'| \to |K|$. K' has additional vertices at $1/4$ and $3/4$ and four edges $\langle 0, 1/4 \rangle$, $\langle 1/4, 1/2 \rangle$, $\langle 1/2, 3/4 \rangle$ and $\langle 3/4, 1 \rangle$.
To try and construct a simplicial approximation to f consider which simplices in K are the carriers of $f(x)$ for points $x \in [0, 1]$. As observed in 5.18,

the carrier of $f(0) = 0$ is $\langle 0\rangle$ and the carrier of $f(1) = 1$ is $\langle 1\rangle$ so that, for a simplicial approximation, we must have $\phi(0) = 0$ and $\phi(1) = 1$. It is also true that $f(1/\sqrt{2}) = 1/2$ which has the carrier $\langle 1/2\rangle$ and so $|\phi|(1/\sqrt{2}) = 1/2$. Since $1/\sqrt{2} \in \langle 1/2, 3/4\rangle$ this means that $1/2 \in |\phi|(\langle 1/2, 3/4\rangle)$.
This suggests trying the following admissible vertex map $\phi(0) = 0$, $\phi(1/4) = \phi(1/2) = \phi(3/4) = 1/2$, $\phi(1) = 1$ for which $|\phi|(\langle 1/2, 3/4\rangle) = \langle 1/2\rangle$. For this admissible vertex map, $|\phi|\colon |K'| \to |L|$ is given by

$$|\phi|(x) = \begin{cases} 2x & \text{for } 0 \leqslant x \leqslant 1/4, \\ 1/2 & \text{for } 1/4 \leqslant x \leqslant 3/4, \\ 2x-1 & \text{for } 3/4 \leqslant x \leqslant 1. \end{cases}$$

You can now easily check that $|\phi|\colon |K'| \to |K|$ is a simplicial approximation to f with respect to K' and K.

The following result gives a criterion for the existence of a simplicial approximation. For this we need the notion of a *star*. For a given simplicial complex K we define the (open) star at a vertex v by

$$\operatorname{star}(v) := \bigcup_{v\in\sigma} \sigma^\circ.$$

5.25 Proposition. Let $f\colon |K| \to |L|$ be a continuous function between the underlying spaces of simplicial complexes K and L. Assume that the image of every star in K is contained in a star of L. Then there is a simplicial approximation of f.

Proof. By the precondition for ever $v \in V(K)$ there exists a vertex $w \in V(L)$ with $f(\operatorname{star}(v)) \subset \operatorname{star}(w) \subset |L|$. We define a vertex map $\phi : V(K) \to V(L)$ by setting $\phi(v) = w$.
Now, given a point x in the interior of a simplex $\langle v_0, \dots, v_r\rangle$ of K. Then x it is contained in $\bigcap_{i=0}^r \operatorname{star}(v_i)$ and, hence,

$$f(x) \in f\left(\bigcap_{i=0}^r \operatorname{star}(v_i)\right) \subset \bigcap_{i=0}^r (\operatorname{star}(\phi(v_i))$$

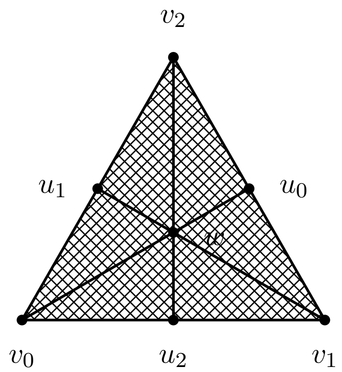
In particular, $\bigcap_{i=0}^r(\operatorname{star}(\phi(v_i))$ is non-empty. All the stars are disjoint unions of interiors of simplices. Hence, the intersection is also a disjoint union of such interiors. If the interior of σ is contained in $\bigcap_{i=0}^r(\operatorname{star}(\phi(v_i))$ then $\phi(v_0), \dots, \phi(v_r)$ have to be vertices of σ. On the one hand this implies that $\langle\phi(v_0), \dots, \phi(v_r)\rangle$ is a face of σ. In particular, $\langle\phi(v_0), \dots, \phi(v_r)\rangle$ is a simplex of L. Hence, ϕ is admissible. On the other hand the carrier of every point in $\bigcap_{i=0}^r(\operatorname{star}(\phi(v_i))$ contains $\langle\phi(v_0), \dots, \phi(v_r)\rangle$ and hence also $|\phi|(x)$. $\square$

5.26 Theorem (The Simplicial Approximation Theorem). Let $f\colon |K| \to |L|$ be a continuous function between the underlying spaces of simplicial complexes K and L. Then, for sufficiently large m, there is a simplicial approximation $|\phi|\colon |K| = |K^{(m)}| \to |L|$ to f with respect to $K^{(m)}$ and L.

Proof. One observes that taking the m-th barycentric subdivision of a simplicial complex of dimension d reduces the maximal diameter of a star by the factor $\left(\frac{d}{d+1}\right)^m$ (Excercise). Now, $\{f^{-1}(\mathrm{star}(w))\}_{w \in V(L)}$ forms an open covering of $|K|$. Hence, by the *Lebesgue Number Lemma* the condition of Proposition 5.25 will be fulfilled for $m \gg 0$. $\square$

5.27 Proposition. Suppose that $K^{(m)}$ is obtained from K by repeated barycentric subdivision, the there is an isomorphism of the simplicial homology groups $\chi_*\colon H_r(K) \to H_r(K^{(m)})$.

Outline proof. We indicate how the result is proved in the case $m = 1$ and then the general result follows by induction on m.
We may define a homomorphism $\chi_r\colon C_r(K) \to C_r(K')$ on generators by mapping each r-simplex of K to the sum of the r-simplices of K' into which it is divided (oriented compatibly). For example, if $\sigma = \langle v_0, v_1, v_2\rangle$ then the first barycentric subdivision of this simplex is as follows.



Here, for example, $w = \widehat{\langle v_0, v_1, v_2\rangle}$ and $u_0 = \widehat{\langle v_1, v_2\rangle}$.
Then

$$\chi_r(\sigma) = \langle v_0, u_2, w\rangle - \langle v_0, u_1, w\rangle - \langle v_1, u_2, w\rangle + \langle v_1, u_0, w\rangle + \langle v_2, u_1, w\rangle - \langle v_2, u_0, w\rangle.$$

It can then be shown that the diagram

$$\begin{tikzcd} C_r(K) \arrow[r, "\chi_r"] \arrow[d, "d_r"] & C_r(L) \arrow[d, "d_r"] \\ C_{r-1}(K) \arrow[r, "\chi_{r-1}"] & C_{r-1}(L) \end{tikzcd} \tag{2}$$

commutes and so (as in the proof of Proposition 5.4) induces homoomorphisms

$$\chi_r\colon Z_r(K) \to Z_r(K'),$$

$$\chi_r \colon B_r(K) \to B_r(K')$$

and so a homomorphism

$$\chi_* \colon H_r(K) \to H_r(K').$$

This homomorphism can be shown to be an isomorphism. Indeed, an inverse comes from an admissible vertex map α, where α sends a barycenter $\hat{\sigma}$ to one chosen vertex of σ. $\square$

5.28 Remark. Note, that the simplicial map $|\alpha| : |K'| \to |K|$ in the proof above is a simplicial approximation of the identity.

5.29 Theorem. Suppose that $f \colon |K| \to |L|$ is a continuous function between the underlying spaces of simplicial complexes. Then, by the Simplicial Approximation Theorem, it has a simplicial approximation $|\phi| \colon |K^{(m)}| \to |L|$ for some repeated barycentric subdivision of K. We define the induced homomorphism

$$f_* \colon H_r(K) \to H_r(L)$$

to be the composition

$$H_r(K) \xrightarrow{\chi_*^m} H_r(K^{(m)}) \xrightarrow{\phi_*} H_r(L).$$

This homomorphism is well-defined and does not depend on the choice of simplicial approximation for f.

5.30 Theorem (Functorial Properties of Homology). (a) If $f \colon |K| \to |K|$ is the identity map, then each hommorphism $f_* \colon H_r(K) \to H_r(K)$ is the identity homomorphism.

(b) Given continuous functions $f \colon |K| \to |L|$ and $g \colon |L| \to |M|$ between the underlying spaces of simplicial complexes then $(g \circ f)_* = g_* \circ f_* \colon H_r(K) \to H_r(M)$ for all r.

5.31 Corollary. The homology groups are topological invariants.

Proof. Consider a homeomorphism $f : |K| \to |L|$. Then functoriality implies $f_* \circ (f^{-1})_* = (f \circ f^{-1})_* = (\mathrm{id}_{|K|})_* = \mathrm{id}_{H_k(|K|)}$ and similarly $f^{-1} \circ f_* = \mathrm{id}_{H_k(|L|}$. Hence, $(f^{-1})_*$ is an inverse of f_* and so $H_k(|L|) \cong H_k(|K|)$. $\square$

5.32 Definition. A simplicial complex K is called *acyclic* if it has the same homology groups as a single vertex, i.e. $H_k(K) \cong H_k(\{v\})$.

5.33 Definition. A *chain homotopy* between chain maps $\psi_k, \phi_k : C_k(K) \to C_k(L)$ is a sequence of homomorophisms $h_k : C_k(K) \to C_{k+1}(L)$, such that

$$d_{k+1} \circ h_k + h_{k-1} \circ d_k = \psi_k - \phi_k.$$

In this situation ψ_* and ϕ_* are called *chain homotopic* to each other.

5.34 Lemma. Two chain homotopic maps ψ_k and ϕ_k induce the same homomorphism on the homology.

Proof. For $z \in Z_k(K)$ we have

$$\psi_k(z) - \phi_k(z) = d_{k+1}(h_k(z)) + h_{k-1}(d_k(z)) = d_{k+1}(h_k(z)) \in B_k(L).$$

Hence, $\psi_*(z)$ and $\phi_*(z)$ are homologous and give rise to the same element of homology. □

5.35 Proposition. $\bar{\Delta}^r$ is acyclic.

Proof. Assume $\Delta^r = \langle e_0, \ldots, e_r \rangle$ and $\Delta^{r-1} = \langle e_0, \ldots, e_{k-1} \rangle$. Then we consider the homomorphisms $h_{k-1} : C_{k-1}(\bar{\Delta}^{r-1}) \to C_k(\bar{\Delta}^r)$ obtained by sending $\langle e_{i_0}, \ldots, e_{i_k} \rangle$ with $0 \leqslant i_j \leqslant r-1$ to $\langle e_{i_0}, \ldots, e_{i_k}, e_r \rangle$ and for $k \geqslant 1$ we obtain

$$d_{k+1}(h_k(\sigma)) = x - h_{k-1}(d_k(\sigma)) \tag{3}$$

and for $k = 0$

$$d_1(h_0(\sigma)) = \sigma - \langle e_r \rangle. \tag{4}$$

On the other hand, the k-simplices of $\bar{\Delta}^r$ either contain e_r and, hence, are of the form $h_{k-1}(\tau)$ with τ an $(k-1)$-simplex in $\bar{\Delta}^{r-1}$ or they do not contain e_r and, hence, are also k-simplex in $\bar{\Delta}^{r-1}$. Hence, every chain $z \in C_k(\bar{\Delta}^r)$ can be written uniquely in the form $z = x + h_{k-1}(y)$ for $k \geqslant 0$ and $z = x + \lambda \cdot \langle v_r \rangle$ for $k = 0$. Assume $d_k(z) = 0$. Then by (3) and (4) we have

$$(d_k(x) + y) - h_{k-2}(d_{k-1}(y)) = 0, \quad \text{ or } \quad (d_k(x) + y) - \lambda \langle v_r \rangle = 0.$$

Because of the uniqueness of the decompositions we have $d_k(x) + y = 0$ in both cases. From (3) we obtain

$$d_{k+1}(h_k(x)) = x - h_{k-1}(d_k(x)) = x + h_{k-1}(y) = z.$$

Hence, z is a boundary. □

5.36 Proposition (on acyclic supports). Consider two simplicial maps and the corresponding chain maps $\phi_k, \psi_k : C_k(K) \to C_k(L)$. Suppose that to every simplex $\sigma \in K$ there is a subcomplex $L(\sigma) \subset L$ such that

(a) if $\sigma' \subset \sigma$ then $L(\sigma') \subset L(\sigma)$;

(b) $L(\sigma)$ is acyclic;

(c) For a k-simplex σ. The complex $L(\sigma)$ contains the support of $\psi_k(\sigma)$ and $\phi_k(\sigma)$.

Then ψ_k and ϕ_k are chain homotopic.

Proof. We construct a chain homotopy inductively. Suppose $k = 0$. Hence, we consider a vertex v in K now $L(v)$ contains $u = \psi(v)$ and $w = \phi(v)$. Since $L(v)$ is acyclic it follows that $L(v)$ is connected. Hence, there is a path of (oriented) edges $\langle u = u_0, u_1\rangle, \langle u_1, u_2\rangle \ldots, \langle u_{\ell-1}, u_\ell = w\rangle$. Now we set $h_0(v) = \sum_{i=1}^{\ell}\langle u_{i-1}, u_i\rangle$.
Suppose we constructed $h_0, \ldots, h_{k-1}$, such that for every i-simplex σ the image $h_i(\sigma)$ is supported in $L(\sigma)$. Consider a k-simplex $\sigma \in K$. Set $c_k = \psi_k(\sigma) - \phi_k(\sigma) - h_{k-1}(d_k(\sigma))$. The we just have to find a chain $h_k(\sigma)$ such that $d_{k+1}(h_k(\sigma)) = c_k$. All simplices of $d_k(\sigma)$ are contained in σ. Hence, $h_{k-1}(d_k(\sigma))$ is supported in $L(\sigma)$. Thus $L(\sigma)$ is a support of c_k.

$$\begin{aligned}
d_k(c_k) &= (d_k \circ \psi_k - d_k \circ \phi_k - d_k \circ h_{k-1} \circ d_k)(\sigma)\\
&= (d_k \circ \psi_k - d_k \circ \phi_k - (\psi_{k-1} \circ d_k - \phi_{k-1} \circ d_k - h_{k-2} \circ d_{k-1} \circ d_{k-2})(\sigma)\\
&= 0
\end{aligned}$$

We see that c_k is a cycle. Since, $k > 0$ and $L(\sigma)$ is acyclic we must have $H_k(L(\sigma)) = 0$. Hence, c_k is also a boundary and there must be some element $h_k(\sigma)$ with $d_k(h_k(\sigma)) = c_k$. □

Proof of Theorem 5.29. We have to show that the induced homomorphisms $\phi_* \circ \chi_*^m = \psi_* \circ \chi_*^n$ for two simplicial approximations ϕ and ψ of f. Let us first assume, that $m = n$. Now, for a simplex $\sigma = \langle v_0, \ldots, v_r\rangle$. Consider x in the interior of σ. Then $\langle\psi(v_0), \ldots, \psi(v_r)\rangle$ and $\langle\phi(v_0), \ldots, \phi(v_r)\rangle$ are both contained in the carrier Δ of $f(x)$, since by the definition of a simplicial approximation we have $|\psi|(x), |\phi|(x) \in \Delta$. Since $\bar{\Delta}$ is acyclic we see that the chain maps $\psi_* = \phi_*$ by Proposition 5.36.
Let us now assume $n > m$. We may replace K by $K^{(m)}$ and n by $p = n - m$. Then we have to prove $\phi_* = \psi_* \circ \chi_*^p$. But this is equivalent to $\phi_* \circ \alpha_*^p = \psi_*$, where α is the simplicial map, which sends $\hat{\sigma}$ to one of the vertices of σ and which on the level of homology gave rise to an inverse of χ_*. Now, since ϕ is a simplicial approximation of f we have $\phi(\mathrm{carr}_K(x)) \subset \mathrm{carr}_L(f(x))$. Note, that α sends the carrier of x in K' to the carrier of x in K. By induction the same is true for α^p. Hence, $\phi \circ \alpha^p(\mathrm{carr}_{K^{(p)}}(x)) \subset \mathrm{carr}_L(f(x))$ and $\phi \circ \alpha^p$ is another simplicial approximation from $|K^{(m)}| \to |L|$ (as ψ). Hence, $\phi_* \circ \alpha_*^p = \psi_*$ for the induced homomorphisms on $H_r(K^{(m)}) \to H_r(L)$. □

5.37 Theorem. If $f_0 \simeq f_1 \colon |K| \to |M|$ are homotopic functions between the underlying spaces of simplicial complexes then $(f_0)_* = (f_1)_* \colon H_r(K) \to H_r(M)$ for all r.

Proof. Assume $H \colon |K| \times I \to |M|$ is a homotopy between f_0 and f_1, i.e. $H(x, i) = f_i(x)$ for $i = 0, 1$. Note, that there is an natural triangulation L of $|K| \times I$, which contains $K \times \{0\}$ and $K \times \{1\}$ as subcomplexes.

$$\begin{aligned}
&\{\langle(v_0, 0), \ldots, (v_i, 0), (v_{i+1}, 1) \ldots (v_r, 1)\rangle \mid \langle v_0 \ldots, v_r\rangle \in K\}\\
\cup&\{\langle(v_0, 0), \ldots, (v_i, 0), (v_i, 1)(v_{i+1}, 1) \ldots (v_r, 1)\rangle \mid \langle v_0 \ldots, v_r\rangle \in K\}
\end{aligned}$$

We have the simplicial inclusion maps

$$i_0\colon |K| \to |K| \times \{0\} \subset |L|, \quad i_1\colon |K| \to |K| \times \{1\} \subset |L|.$$

For $\sigma \in K$ consider the subcomplex $L(\sigma) \subset L$ with $|L(\sigma)| = \sigma \times I$. Then

$$i_0(\sigma) = \sigma \times \{0\} \subset L(\sigma) \supset \sigma \times \{0\} = i_1(\sigma).$$

Moreover, for every r-simplex σ the subcomplex $L(\sigma)$ is acyclic, since $|L(\sigma)| = \sigma \times I \cong \Delta^k$. Hence, by Proposition 5.36 $(i_0)_* = (i_1)_*$ and consequently

$$(f_0)_* = (H \circ i_0)_* = H_* \circ (i_0)_* = H_* \circ (i_1)_* = (H \circ i_1)_* = (f_1)_*.$$

$\square$

5.38 Corollary. The homology groups are a homotopy invariant, i.e. if $f\colon |K| \to |L|$ is a homotopy equivalence of the underlying spaces of two simplicial complexes then $f_*\colon H_r(K) \to H_r(L)$ is an isomorphism for all r.

Proof. Suppose that $g\colon |L| \to |K|$ is a homotopy inverse for f. Then

$$g_* \circ f_* = (g \circ f)_* \text{ by 5.30(a) } = (\mathrm{id}_{|K|})_* \text{ by 5.37 } = \mathrm{id}_{H_r(K)} \text{ by 5.30(b)}$$

and similarly $f_* \circ g_* = \mathrm{id}_{H_r(L)}$. Hence $f_*\colon H_r(K) \to H_r(L)$ is an isomorphism for each r with inverse g_*. $\square$

5.39 Definition. If a topological space X is homotopy equivalent to the underlying space of a simplicial complex $|K|$ we may define the homology groups of X by $H_r(X) = H_r(K)$.
Notice that these groups are well-defined up to isomorphism since, if $h\colon |K| \to X$ and $k\colon |L| \to X$ are two homotopy equivalences from the underlying spaces of simplicial ocmplexes K and L, then $k^{-1} \circ h\colon |K| \to |L|$ is a homotopy equivalence and so, by Corollary 5.38, induces isomorphism $H_r(K) \to H_r(L)$ for all r.

Homology of spheres and degree of selfmaps on spheres

5.40 Theorem. The homology groups of the n-sphere S^n for $n \geqslant 1$ are given by

$$H_r(S^n) \cong \begin{cases} \mathbb{Z} & \text{for } r = 0 \text{ and } r = n, \\ 0 & \text{otherwise.} \end{cases}$$

To prove this first of all notice the following result.

5.41 Proposition. If the underlying space $|K|$ of a simplicial complex K is contractible then

$$H_r(K) \cong \begin{cases} \mathbb{Z} & \text{for } r = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. First all notice that if K is the simplicial complex consisting of one vertex and no other non-empty simplices than this has the homology groups stated in the proposition since $C_0(K) \cong \mathbb{Z}$ but all the other chain groups are trivial and so all of the boundary homomorphisms are trivial.
The result now follows from Corollary 5.38. $\square$

Proof of Theorem 5.40. First of all notice that since Δ^{n+1} is a convex subset of $\mathbb{R}^{n+2}$ it is contractible [Exercise]. Hence the homology groups of the simplicial complex $\bar{\Delta}^{n+1}$ are given by Proposition 5.41.
Now recall that the underlying space of the n-skeleton $K = (\bar{\Delta}^{n+1})^{[n]}$ is homeomorphic to S^n (Example 4.8(c)).
Since K is n-dimensional it has trivial homology groups in dimensions above n. In dimensions $0 \leqslant r \leqslant n$, $C_r(K) = C_r(\bar{\Delta}^{n+1})$ with the same boundary homomorphims between these groups. Hence in dimensions $0 \leqslant r \leqslant n-1$, $H_r(K) = H_r(\bar{\Delta}^{n+1})$. However, in dimension n, $B_n(K) = 0$ since $C_{n+1}(K) = 0$ and so $H_n(K) = Z_n(K)$ a free group of rank $\beta_n(K)$, the nth Betti number of K.
By Problems 4, Question 5(ii), the Euler characteristic of K is given by $\chi(K) = 1 + (-1)^n$. But $\beta_0(K) = 1$ and $\beta_r(K) = 0$ for all $r \neq 0$ and n since the homology groups are trivial in these dimensions. Hence, by Theorem 4.37, $\beta_n(K) = 1$. Hence $H_n(K) \cong \mathbb{Z}$ as required to complete the proof. $\square$

5.42 Definition. Suppose that $h\colon |K| \to S^n$ is a triangulation of the n-sphere (for example $K = (\bar{\Delta}^{n+1})^{[n]}$. Given a continuous function $f\colon S^n \to S^n$ then this induces a continuous function $h^{-1} \circ f \circ h\colon |K| \to |K|$ which in turn induces a homomorphism $(h^{-1} \circ f \circ h)_*\colon H_n(K) \to H_n(K)$ which must be given by $[z] \mapsto \lambda[z]$ for $\lambda \in \mathbb{Z}$ since $H_n(K) \cong \mathbb{Z}$. The integer $\lambda \in \mathbb{Z}$ is called the *degree* of the continuous function f denoted $\deg(f)$.

5.43 Proposition. The degree of a continuous function $f\colon S^n \to S^n$ is well-defined and does not depend on the choice of triangulation $h\colon |K| \to S^n$.

Proof. Suppose that $h\colon |K| \to S^n$ is a triangulation of S^n and that using this triangulation the degree of a continuous function $f\colon S^n \to S^n$ has degree $\deg(f) = \lambda$.
Suppose that $k\colon |L| \to S^n$ is a second triangulation of S^n. Then $g = h^{-1} \circ k\colon |L| \to |K|$ is a homeomorphism and so induces an isomorphism $g_*\colon H_n(L) \to H_n(K)$.
Now for $[w] \in H_n(L)$, using the following commutative diagram,

$$\begin{tikzcd}
{|K|} \arrow[r, "h"] & S^n \arrow[r, "f"'] & S^n \arrow[r, "h^{-1}"] & {|K|} \\
{|L|} \arrow[u, "g"] \arrow[r, "k"] & S^n \arrow[u, "\mathrm{id}"'] \arrow[r, "f"] & S^n \arrow[u, "\mathrm{id}"'] \arrow[r, "k^{-1}"] & {|L|} \arrow[u, "g"']
\end{tikzcd}$$

$$
\begin{aligned}
(k^{-1} \circ f \circ k)_*([w]) &= (g^{-1} \circ h^{-1} \circ f \circ h \circ g)_*([w]) \\
&= \big((g^{-1})_* \circ (h^{-1} \circ f \circ h)_* \circ g_*\big)([w]) \\
&= (g^{-1})_*\big((h^{-1} \circ f \circ h)_*\big(g_*([w])\big)\big) \\
&= (g^{-1})_*\big(\lambda g_*([w])\big) \\
&= \lambda (g^{-1})_* \circ g_*([w]) \\
&= \lambda [w]
\end{aligned}
$$

showing that using that using k in place of h gives the same value for the degree. □

5.44 Proposition.

(a) $f \simeq g \colon S^n \to S^n \Rightarrow \deg(f) = \deg(g)$.

(b) The identity function $I \colon S^n \to S^n$ has degree 1.

(c) A constant function $c \colon S^n \to S^n$ has degree 0,

Proof. (a) This follows from Theorem 5.37

(b) This follows from Theorem 5.30(a).

(c) A constant function $c \colon S^n \to S^n$ factors through a one-point space $\{a\}$: $S^n \to \{a\} \to S^n$ and so, given a triangulation $h \colon |K| \to S^n$, the homomorphism of H_n induced by $h^{-1} \circ f \circ h \colon |K| \to |K|$, by Theorem 5.29, factors through $H_n(\{a\}) = 0$ by Proposition 5.41. Hence $(h^{-1} \circ f \circ h)_*([z]) = 0$ giving the result.

□

5.45 Example. We want to calculcate the degree of $f : S^1 \to S^1 \subset \mathbb{C}$ given by $f(z) = z^2$. We consider the triangulation K and simplicial approximation g from Problems 6 Question 4 which lives on K'. Note, that a generator of $H^n(K)$ is given by $z = [\langle v_0, v_1\rangle + \langle v_1, v_2\rangle + \langle v_2, v_0\rangle]$. Then we have

$$\chi_*([z]) = [\langle v_0, w_0\rangle + \langle w_0, v_1\rangle + \langle v_1, w_1\rangle + \langle w_1, v_2\rangle + \langle v_2, w_2\rangle + \langle w_2, v_0\rangle]$$

Now,

$$f_*([z] = g_*(\chi_*([z])) = [\langle v_0, v_1\rangle + \langle v_1, v_2\rangle + \langle v_2, v_0\rangle + \langle v_0, v_1\rangle + \langle v_1, v_2\rangle + \langle v_2, v_0\rangle] = 2[z].$$

Hence, $\deg(f) = 2$.

5.46 Proposition. The antipodal map $\tau \colon S^n \to S^n$ given by $\tau(\mathbf{x}) = -\mathbf{x}$ has degree $(-1)^{n+1}$.

Proof. We first of all define a convenient triangulation of S^n.
For $1 \leqslant i \leqslant n+1$, let $v_i = \varepsilon_i$, the standard ith basis vector of $\mathbb{R}^{n+1}$ and $v_{-i} = -\varepsilon_i$. Let

$$K = \{ \langle v_{i_1}, v_{i_2}, \ldots, v_{i_r} \rangle \mid 1 \leqslant |i_1| < |i_2| < \ldots < |i_r| \leqslant n+1,\ 0 \leqslant r \leqslant n+1 \}.$$

Then K is a simplicial complex and $|K| = \{ \mathbf{x} \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} |x_i| = 1 \}$.
Radial projection gives a homeomorphism $h \colon |K| \to S^n$.
A generator for $H_n(K) = Z_n(K) \cong \mathbb{Z}$ is given by

$$z = \langle v_1, v_2, \ldots, v_{n+1} \rangle - \langle v_{-1}, v_2, \ldots, v_{n+1} \rangle + \ldots + (-1)^{n+1} \langle v_{-1}, v_{-2}, \ldots, v_{-(n+1)} \rangle$$

where the coefficient is $(-1)^r$ if there are r vertices of the form v_{-i}.
Let $\tau \colon S^n \to S^n$ be the antipodal function $\tau(\mathbf{x}) = -\mathbf{x}$. Then $h^{-1} \circ \tau \circ h \colon |K| \to |K|$ is a simplicial map coming from the admissible vertex map $\phi(v_i) = v_{-i}$, $\phi(v_{-i}) = v_i$.
But then $\phi_*(z) = (-1)^{n+1} z$. Hence by the definition of degree $\deg(\tau) = (-1)^{n+1}$. $\square$