Autumn Semester 2017–2018

MATH41071/MATH61071 *Algebraic topology*

Problems 1: Topological Surfaces

1. Prove that, if M_1 is a topological n_1-manifold and M_2 is a topological n_1-manifold then the product space $M_1 \times M_2$ is a topological (n_1+n_2)-manifold (Proposition 1.4).

2 (1). Prove that a compact locally Euclidean topological space is second countable.

3. Define an equivalence relation on the product space $\mathbb{R}^2 \times \{0, 1\}$ (where $\mathbb{R}^2$ has the usual topology and $\{0, 1\}$ has the discrete topology by $(\mathbf{x}, 0) \sim (\mathbf{x}, 1)$ when $\mathbf{x} \neq \mathbf{0}$. Prove that the identification space is locally Euclidean but not Hausdorff.

4. Show that actually two charts are enough to give an atlas for S^2 (actually for every S^n) by using stereographic projection.

5 (1). Show that the closed disc D^2 is not a topological surface.

6. Let $\sim$ be the equivalence relation on I^2 (where $I = [0, 1]$ with the usual topology) generated by $(x, 0) \sim (x, 1)$, $(0, y) \sim (1, y)$ for $x \in I$, $y \in I$. Prove that the quotient space $I^2/{\sim}$ is homeomorphic to the $S^1 \times S^1$, the torus defined as a product space.

7 (1). Define an equivalence relation on $D^2 = \{\, \mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| \leqslant 1 \,\}$ (with the usual topology) by

$$(x, y) \sim (x', y') \Leftrightarrow (x, y) = (x', y') \text{ or } [(x, y) \text{ and } (x', y') \in S^1,\ x' = x \text{ and } y' = -y].$$

Prove that $D^2/{\sim}$ is homeomorphic to the 2-sphere $S^2 = \{\, \mathbf{x} \in \mathbb{R}^3 \mid |\mathbf{x}| = 1 \,\}$.

8 (2). Recall that the Klein bottle is the identification space $I^2/{\sim}$ where the equivalence relation is generated by $(s, 0) \sim (s, 1)$ and $(0, t) \sim (1, 1-t)$. Using cut and paste methods, prove that the space obtained by identifying the boundary circles of two Möbius bands is homeomorphic to the Klein bottle. Deduce that, in the classification theorem, the Klein bottle is homeomorphic to P_2.

9. Using cut and past methods, outine a prove that, for any topological surface S, the connected sum with the Klein bottle, $S\#K$, is homeomorphic to S with a handle attached. [Hint: Mimic the proof of Proposition 1.18.]

10. Explain why $P^2 \# T_1$ is homeomorphic to $P^2 \# K$ where K is the Klein bottle so that, in the classification theorem, $T_1 \# P_1 \cong P_3$. [Hint: Both spaces are obtained by attaching a handle to P^2. Is there any difference in the way these handles are attached?]

11 (2). Make two Möbius bands.

- Cut the first one along the central circle parallel to the boundary of the band.
- Cut the the second one, again parallel to the boundary, but this time along a non-central circle.

Reproduce the results via cut and paste methods using the representation of the Möbius band as the unit square with identified edges.

12 (fun)**.** Build the Klein bottle paper model, but stop before doing the last gluing step, i.e. "closing the bottle". Verify that it is glued out of two Möbius bands.