Autumn Semester 2017–2018

MATH41071/MATH61071 *Algebraic topology*

Solutions 1

1. First of all recall that a product of Hausdorff spaces is Hausdorff (Proposition T.27) and a product of second countable spaces is second countable (Proposition T.28). For locally Euclidean suppose that $(x_1, x_2) \in M_1 \times M_2$. Then, since M_i is a topological n_i-manifold there is a homeomorphism (a chart) $\phi_i \colon U_i \to V_i$ where U_i is open in $\mathbb{R}^{n_i}$ and $x \in V$ which is open in M_i. Hence, $\phi_1 \times \phi_2 \colon U_1 \times U_2 \to V_1 \times V_2$ is the required chart around (x_1, x_2).

2. Let $\{\, \phi_\lambda \colon U_\lambda \to V_\lambda \mid \lambda \in \Lambda \,\}$ be an atlas on the compact local Euclidean space X. Then $\{\, V_\lambda \mid \lambda \in \Lambda \,\}$ is an open cover of X and so, since X is compact this has a finite subcovering $\{\, V_{\lambda_1}, V_{\lambda_2}, \ldots, V_{\lambda_n} \,\}$. Then each V_{λ_i} is homeomorphic to an open subspace in some $\mathbb{R}^n$ and so has a countable basis. The sets in each of these bases are all open in X since V_{λ_i} is open in X. The union of these bases is countable and gives a basis for X since any open set $V \subset X$ can be written $V = \bigcup_{i=1}^n (V \cap V_{\lambda_i})$ and each $V \cap V_{\lambda_i}$ is open in V_{λ_i} and so may be written as a union of basic open sets. Hence X is second countable.

3. An atlas for the identification space is given by the two maps $\phi_i \colon \mathbb{R}^2 \to (\mathbb{R}^2 \times \{0,1\})/{\sim}$ given by $\phi_i(\mathbf{x}) = [(\mathbf{x}, i)]$ for $i = 0,\ 1$. So the identification space is locally Euclidean.
To see that the space is not Hausdorff suppose that U_0 and U_1 are open subsets of the identification space such that $[(\mathbf{0}, 0)] \in U_0$, $[(\mathbf{0}, 1)] \in U_1$, then, for $i = 0,\ 1$, $q^{-1}(U_i)$ is an open set in $\mathbb{R}^2 \times \{0,1\}$ containing $(\mathbf{0}, i)$ so that there is an $\varepsilon_i > 0$ such that $B_{\varepsilon_i}(\mathbf{0}) \times \{i\} \subset q^{-1}(U_i)$. Now, if $\mathbf{x}$ is a vector such that $0 < |\mathbf{x}| < \min(\varepsilon_1, \varepsilon_2)$, $[(\mathbf{x}, 0)] = [(\mathbf{x}, 1)] \in U_1 \cap U_2$ and so the disjoint open sets whose existence is required by the Hausdorff condition do not exist. Hence the identification space is not Hausdorff.

4. We may choose the two open sets $V^+ = S^2 \setminus \{(0,0,1)\}$ and $V^- = S^2 \setminus \{(0,0,-1)\}$. Now, the map

$$\phi^{\pm} : \mathbb{R}^2 \to V^{\pm}; (u_1, u_2) \mapsto \left(\frac{2u_1}{1 + u_1^2 + u_2^2}, \frac{2u_2}{1 + u_1^2 + u_2^2}, \pm \frac{u_1^2 + u_2^2 - 1}{1 + u_1^2 + u_2^2} \right)$$

is continuous and has an inverse (check)

$$(\phi^{\pm})^{-1} : V^{\pm} \to \mathbb{R}^2; (x_1, x_2, x_3) \mapsto \left(\frac{x_1}{1 \mp x_3}, \frac{x_2}{1 \mp x_3} \right)$$

which is continuous on $V^{\pm}$, since $x_3 \neq \pm 1$.

5. Pick a point x on the boundary of the closed unit disc. Assume we have an open neighbourhood V of x which is homeomorphic to an open subset of $\mathbb{R}^2$. Then for $\epsilon > 0$ sufficiently small we can find an open disc $B_\epsilon(x)$ such that $B := (B_\epsilon(x) \cap D^2) \subset V$. Now, restricting the homeomorphism $\phi : V \to U \subset \mathbb{R}^2$ to B we obtain an homeomorphism between B and the open subset $U' := \phi(B) \subset \mathbb{R}^2$ and, hence, a homeomorphism between $B\setminus\{x\}$ and $U' \setminus \{x'\}$, where $x' := \phi(x)$.
Observe that $B\setminus\{x\}$ is a convex subset of $\mathbb{R}^2$, indeed B is (it's an intersection of convex sets) and x doesn't lie on any line segment inside B. Hence $\pi_1(B \setminus \{x\}, y_0)$ is trivial for every choice of y_0 by Example 6.3 (b) in the *Topology* notes.
Now, choose some open disc $B_\delta(x') \subset U'$ then the boundary circle C of $B_{\delta/2}(x')$ is a retract of $U' \setminus \{x'\}$. Indeed, $r : u \mapsto \frac{\delta}{2|u-x'|}(u - x') + x'$ is a retraction map.
Take a point $y_0' \in C$ and $y_0 = \phi^{-1}(y_0')$. Because of the functorial properties of the fundamental group we must have

$$0 = \pi_1(B \setminus \{x\}, y_0) \cong \pi_1(U' \setminus \{x'\}, y_0')$$

(6.23 in Peter Eccles' notes). Again by the functorial properties the composition

$$\pi_1(C, y_0') \xrightarrow{i_*} \pi_1(U' \setminus \{x'\}, y_0') \xrightarrow{r_*} \pi_1(C, y_0')$$

has to be the identity on $\pi_1(C, y_0')$ (6.22 in Peter Eccles' notes), but this is impossible, since $\pi_1(U' \setminus \{x'\}, y_0')$ was assumed to be trivial. Note, that this argument is the same as in the proof of Theorem 8.3 in Peter Eccles' notes.

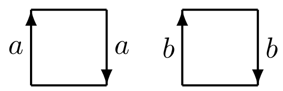
6. Define $f\colon I^2 \to S^1 \times S^1$ by $f(x, y) = (\exp(2\pi ix), \exp(2\pi iy))$. Then $f(x_1, y_1) = f(x_2, y_2) \Leftrightarrow (x_1, y_1) \sim (x_2, y_2)$. So f induces a bijection $F\colon I^2/\!\sim\ \to S^1 \times S^1$ by $F[x, y] = f(x, y)$. This is continuous by the universal property of the quotient topology. I^2 is compact (closed bounded subset of $\mathbb{R}^2$) and so its continuous image $I^2/\!\sim$ is compact. $S^1 \times S^1$ is Hausdorff (Proposition 7.6) and so F is a homeomorphism as required since it is a continuous bijection from a compact space to a Hausdorff space.

7. Notice that $D^2 = \{\, (x, y) \mid -1 \leqslant x \leqslant 1,\ -\sqrt{1-x^2} \leqslant y \leqslant \sqrt{1-x^2}\,\}$.
Define $f\colon D^2 \to S^2$ by

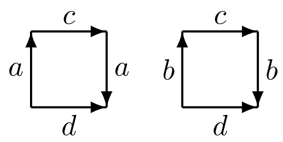
$$f(x, y) = \Big(\sqrt{1-x^2}\cos\big(\pi y/\sqrt{1-x^2}\,\big), \sqrt{1-x^2}\sin\big(\pi y/\sqrt{1-x^2}\,\big), x\Big).$$

Then this is a continuous surjection such that $f(x, y) = f(x', y') \Leftrightarrow (x, y) \sim (x', y')$. Hence, since D^2 is compact and S^2 is Hausdorff, this induces in the usual way a continuous bijection and so a homeomorphism $F\colon D^2/\!\sim\ \to S^2$ by $F([\mathbf{x}]) = f(\mathbf{x})$.

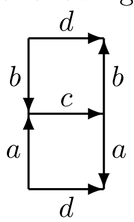
8. Two Möbius bands can be shown as follows.



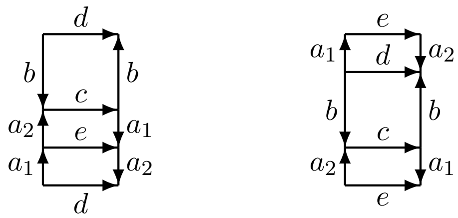
Identifying the boundary circles gives the following.



Gluing along the edge c gives the following.



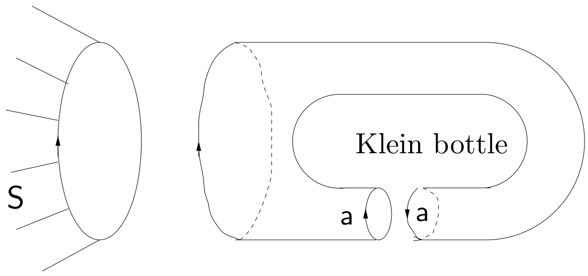
Here the identifications of the sides are not quite as required for the Klein bottle. However, if we cut along the line marked e in the first diagram below and then glue along d we get the second diagram below which we can recognize as the Klein bottle since top and bottom are identified without a twist and the two sides are identified with a twist.

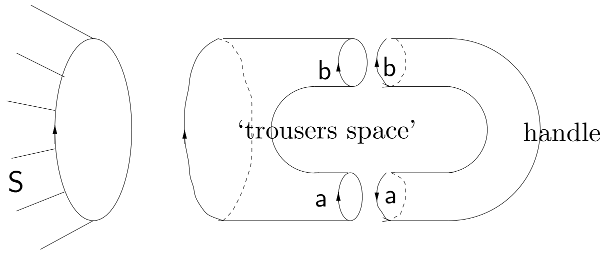


Since the connected sum of two projective planes is obtained by removing a disc from each and gluing the resulting spaces together along the boundary circles and since the result of removing a disc from the projective plane is the Möbius band (Proposition 1.20) it follows that P_2 is homeomorphic to the Klein bottle.

Notice that it is also possible to do this question (using techniques from §2) by reducing the symbol representing the third diagram above to the standard form: $ab^{-1}db^{-1}ad^{-1} \sim aabd^{-1}bd^{-1} \sim aabbdd^{-1} \sim aabb$ which represents P_2. The Klein bottle is given by $ab^{-1}ab \sim aabb$ and so is also P_2. This is not really a different proof since the rules for manipulating symbols are proved by cut and paste techniques.

9 (This argument is very similar to the argument used in lectures in the outline proof of Proposition 1.18. The difference is that whereas the torus is formed by identifying the ends of a cylinder in one way, the Klein bottle is formed by identifying the ends of a cylinder with a twist.)**.** The first picture shows the Klein bottle as a cylinder with the ends to be identified with a disc removed to form the connected sum with S.





Now, as in the proof of Proposition 1.18 in the lectures, we cut the Klein bottle leaving a handle (homeomorphic to the cylinder $S^1 \times I$) and the ‘trousers space’ as above.
Now the proof is completed in the same way as the proof of Proposition 1.18. However, notice that, whereas in Proposition 1.18 the arrows on the two circles bounding the two discs removed from S are in opposite directions, in this case the circles have arrows in the same direction.

10. When we attach a handle to a surface as in Proposition 1.18 and Question 7, if the boundary circles of the cylinder are oriented in the same direction, then the connected sum with the torus is obtained by removing two discs from the surface and gluing the ends of the cylinder to the boundary circles when these are oriented in the opposite directions from each other. On the other hand the connected sum with the Klein bottle is obtained by gluing the ends of the cylinder to the boundary circles when these are oriented in the same direction as each other. However, it the surface contains a Möbius band (as does P^2) then however the boundary circles are oriented if you go one way round the Möbius band the circles will appear to have the same orientation whereas if you go the other way round they will appear to have the oppositive orientation.
It is easier to talk about this solution then to write it down. A more formal proof (but possibly less enlightening) is provided by Corollary 2.27.