Autumn Semester 2017–2018

MATH41071/MATH61071 *Algebraic topology*

Solutions 6

1. Using the simplicial complex K of Solutions 4, Question 3(b) the subcomplex L whose underlying space is the boundary circle of the Möbius band consists of the edges $\langle v_1, v_3\rangle$, $\langle v_3, v_4\rangle$, $\langle v_2, v_4\rangle$, $\langle v_2, v_5\rangle$, $\langle v_5, v_6\rangle$, $\langle v_1, v_6\rangle$ and their vertices. Then $H_1(L) = Z_1(L) \cong \mathbb{Z}$ generated by

$$x = \langle v_1, v_3\rangle + \langle v_3, v_4\rangle - \langle v_2, v_4\rangle + \langle v_2, v_5\rangle + \langle v_5, v_6\rangle - \langle v_1, v_6\rangle.$$

But then, using the notation of Solutions 5, Question 3(v), $i_*(x) = x_1 + x_2$ and so the induced map in homology is given by $i_*(x) = [x_1 + x_2] = [x_1] + [x_2] = 2[x_1]$.

2. There are three conditions for an equivalence relation.

reflexibity: Given a continuous function $f\colon X \to Y$ then $f \simeq f$. A homotopy is given by $H(x,t) = f(x)$.

symmetry: Given homotopic functions $f_0 \simeq f_1\colon X \to Y$ then $f_1 \simeq f_0$. Given a homotopy $H\colon f_0 \simeq f_1$ then a homotopy $K\colon f_1 \simeq f_0$ is given by $K(x,t) = H(x, 1-t)$.

transitivity: Given homotopic functions $f_0 \simeq f_1\colon X \to Y$ and $f_1 \simeq f_2\colon X \to Y$ then $f_0 \simeq f_2\colon X \to Y$. Given homotopies $H\colon f_0 \simeq f_1$ and $K\colon f_1 \simeq f_2$ then a homotopy $L\colon f_0 \simeq f_2$ is given by

$$L(x,t) = \begin{cases} H(x, 2t) & \text{for } 0 \leqslant t \leqslant 1/2, \\ K(x, 2t-1) & \text{for } 1/2 \leqslant t \leqslant 1. \end{cases}$$

This is well-defined since $H(x,1) = f_1(x) = K(x,0)$ and is continuous by the Gluing Lemma.

Hence homotopy is an equivalence relation.

3. (a) Suppose that $X \equiv P$, a one-point space $P = \{a\}$. Let $c\colon X \to P$ be the constant map $c(x) = a$ and $f\colon P \to X$ be an map $f(a) = x_0$ giving a homotopy equivalence. Then $c \circ f = I\colon P \to P$ and, since $f \circ c \simeq I\colon X \to X$ there is a homotopy $H\colon X \times I \to X$ such that $H(x,0) = (f \circ c)(x) = x_0$ and $H(x,1) = x$. Then for each $x \in X$ we may define a path from x_0 to x in X by $\gamma(t) = H(x,t)$. Hence X is path-connected.

(b) Given the notation of (a), the singleton subset $\{x_0\}$ is a deformation retract of X since $H\colon i \circ r \simeq I\colon X \to X$ where i is the inclusion map and $r\colon X \to \{x_0\}$ is the constant map.

Now, for any other point $x_1 \in X$, we may define a homotopy $K\colon i_1 \circ r_1 \simeq I\colon X \to X$ (where $i_1\colon \{x_1\} \to X$ is the inclusion map and $r_1\colon X \to \{x_1\}$ is the constant map) by

$$K(x,t) = \begin{cases} H(x_1, 1-2t) & \text{for } 0 \leqslant t \leqslant 1/2, \\ H(x, 2t-1) & \text{for } 1/2 \leqslant t \leqslant 1. \end{cases}$$

This map is well defined since, for $t = 1/2$, $H(x_1, 0) = x_0 = H(x, 0)$. It is continuous by the Gluing Lemma.

4.

(a) (i) Any simplicial approximation of f must map $0 \mapsto 0$ and $1 \mapsto 1$. The only admissible vertex map $V(K) \to V(L)$ which does this maps $\frac{1}{3} \mapsto \frac{2}{3}$ which does not give a simplicial approximation to f since $|\phi|(\frac{2}{3}) = \frac{5}{6}$ does not lie in the carrier of $f(\frac{2}{3}) = \frac{4}{9}$ which is $[0, \frac{2}{3}]$.

(ii) If we look at simplicial maps $|K'| \to |L|$ (using the first barycentric subdivision of K with vertices at $0, \frac{1}{6}, \frac{1}{3}, \frac{2}{3}$ and 1, we are still out of luck. Since $1 \mapsto 1$ we must have $\frac{2}{3} \mapsto \frac{2}{3}$ or $\frac{2}{3} \mapsto 1$. In the first case, points in the range $\frac{2}{3} < x < \sqrt{\frac{2}{3}}$ fail the approximation condition and in the second case points in the range $\frac{1}{3} < x < \frac{2}{3}$ fail the condition.

(iii) However, using the second barycentric subdivision of K , there is a simplicial map $K'' \to L$ giving a simplicial approximation. For example, send all the vertices of K'' to 0 apart from $\frac{5}{6} \mapsto \frac{2}{3}$ and $1 \mapsto 1$.

(b) We have to take $K^{(2)}$ consisting of the intervals $[0, 1/4]$, $[1/4, 1/2]$, $[1/2, 3/4]$ and $[3/4, 1]$ and their endpoints. Now, one observes that

$$\begin{aligned} \mathrm{star}(0) = [0, 1/4) \subset [0, 4/5) &= f^{-1}(\mathrm{star}(v_5)), \\ \mathrm{star}(1/4) = (0, 1/2) \subset [0, 4/5) &= f^{-1}(\mathrm{star}(v_5)), \\ \mathrm{star}(1/2) = (1/4, 3/4) \subset [0, 4/5) &= f^{-1}(\mathrm{star}(v_5)), \\ \mathrm{star}(3/4) = (1/2, 1) \subset (1/5, 1] &= f^{-1}(\mathrm{star}(v_2)), \\ \mathrm{star}(1) = (3/4, 1] \subset (1/5, 1] &= f^{-1}(\mathrm{star}(v_2)). \end{aligned}$$

Hence, by (a) the vertex map s given by $s(0) = s(1/4) = s(1/2) = v_5$ and $s(3/4) = s(1) = v_2$ defines a simplicial approximation to f.

5. Radial projection gives a homeomorphism $h\colon |K| \to S^1$. Set $v_0 = 1$, $v_1 = e^{\frac{2}{3}\pi i}$ and $v_2 = e^{\frac{4}{3}\pi i}$. Then $f(z) = z^2$ corresponds to a function $g\colon |K| \to |K|$ with $g(v_0) = v_0$, $g(v_1) = v_2$ and $g(v_2) = v_1$. So a simplicial approximation to g would have to be given by this vertex map. However, the simplicial map coming from this admissible vertex map is $z \mapsto \bar{z} = z^{-1}$ which is not homotopic to $z \mapsto z^2$. You can easily find points where the simplicial approximation condition fails, e.g. $g(h^{-1}(-1) = h(f(-1)) = 1$ whose carrier is $\langle v_0 \rangle$, but $g(h^{-1}(-1)) = h(-1) \notin \langle v_0 \rangle$.

The first barycentric subdivision introduces new vertices at $w_o = h^{-1}\big(e^{\pi i/3}\big)$, $w_1 = h^{-1}(-1)$ and $w_2 = h^{-1}\big(e^{5\pi i/3}\big)$. Then the simplicial map corresponding to the admissible vertex map $w_1 \mapsto v_1$, $v_1 \mapsto v_2$, $w_1 \mapsto v_0$, $v_2 \mapsto v_1$, $w_2 \mapsto v_2$ nd $v_0 \mapsto v_0$ actually is the function g and so is certainly a simplicial approximation to it.

6. First of all notice that if a function $f\colon S^m \to S^n$ is not a surjection then it is homotopic to a constant function since, if $v \in S^n$ then $S^n \setminus \{v\} \cong \mathbb{R}^n$. This means that if v is not a value of f then f factors as $\phi^{-1} \circ f_1$ where $\phi\colon S^n \setminus \{v\} \to \mathbb{R}^n$ is a homeomorphism and $f_1 = \phi \circ f\colon S^m \to \mathbb{R}^n$. Then the homotopy $H\colon S^m \times I \to S^n$ defined by $H(x,t) = \phi^{-1}\big(tf_1(x)\big)$ shows that f is homotopic to a constant function.

A homeomorphism $\phi\colon S^n \setminus \{v\} \to \mathbb{R}^n$ is given by stereographic projection from v: we map a point $x \in S^n \setminus \{v\}$ to the point where the line through v and x meets the hyperplane $v^\perp$. Since a general point on this line is given by $x + t(v - x)$ we find the point where this cuts the line by solving $\big((x + t(v - x)\big) \perp v = 0$ which gives $t = (x.v)(v - x)/(1 - x.v)$.

At first sight this might appear to prove the required result since it seems obvious that, if $m < n$, the a continuous function $f\colon S^m \to S^n$ cannot be a surjection. However, this 'obvious' result is in fact false and remarkably you can find continuous surjections $S^m \to S^n$. I don't know a good reference for this general result. However, by using a 'space filling curve' (a continuous surjection $I \to I^2$) you can construct a continuous surjection $S^1 \to S^2$ which demonstrates that the 'obvious' result is false in this case.

We can overcome this problem by use of the Simplicial Approximation Theorem. Let $K = (\bar{\Delta}^{m+1})^{[m]}$ and $L = (\bar{\Delta}^{n+1})^{[n]}$ so that $|K| \cong S^m$ and $|L| \cong S^n$. Then a continuous function $f\colon S^m \to S^n$ gives a continuous function $g\colon |K| \to |L|$ (by composing f with homeomorphisms). By the Simplicial Approximation Theorem, g is homotopic to a simplicial map $|K^{(r)}| \to |L|$ which is not a surjection since $|K^{(r)}|$ will be mapped to the underlying space of the m-skeleton of L. Hence f is homotopic to a function $S^m \to S^n$ which is not a surjection.