INFINITE TRANSITIVITY ON UNIVERSAL TORSORS

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Abstract. Let X be an algebraic variety covered by open charts isomorphic to the affine space and $q : \widehat{X} \to X$ be the universal torsor over X. We prove that the automorphism group of the quasiaffine variety $\widehat{X}$ acts on $\widehat{X}$ infinitely transitively. Also we find wide classes of varieties X admitting such a covering.

Introduction

Universal torsors were introduced by Colliot-Thélène and Sansuc in the framework of arithmetic geometry to investigate rational points on algebraic varieties, see [11], [12], [29]. In the last years they were used to obtain positive results on Manin's Conjecture. Another source of interest is Cox's paper [13], where an explicit description of the universal torsor over a toric variety is given. This approach had an essential impact on toric geometry. For generalizations and relations to Cox rings, see [17], [8], [9], [16], [4].

Let X be a smooth algebraic variety. Assume that the divisor class group $\mathrm{Cl}(X)$ is a lattice of rank r. The universal torsor $q : \widehat{X} \to X$ is a locally trivial H-principal bundle with certain characteristic properties, where H is an algebraic torus of dimension r, see [29, Section 1]; here $\widehat{X}$ is a smooth quasiaffine algebraic variety.

The aim of this paper is to show that under some mild restrictions on X the automorphism group $\mathrm{Aut}(\widehat{X})$ acts on $\widehat{X}$ infinitely transitively. We use a construction of [21] to show that open cylindric subsets on X define one-parameter unipotent subgroups L_i in $\mathrm{Aut}(\widehat{X})$. It turns out that the subgroup generated by L_i acts on $\widehat{X}$ transitively. The next task is to prove that transitivity implies infinite transitivity. To this end, we generalize some results of [5] from affine to quasiaffine case.

The paper is organized as follows. In Section 1 we recall basic definitions and facts on Cox rings and universal torsors. The group of special automorphisms $\mathrm{SAut}(Y)$ of an algebraic variety Y is considered in Section 2. It is shown in [5] that if Y is affine of dimension at least 2 and the group $\mathrm{SAut}(Y)$ acts transitively on an open subset in Y, then this action is infinitely transitive. In Theorem 2 we extend this result to the case when Y is quasiaffine.

It is observed in [21] that open cylindric subsets on a projective variety X give rise to one-parameter unipotent subgroups in the automorphism group of an affine cone over X. This idea is developed further in [22] and [25]. In Section 3 we show that if X is a smooth algebraic variety with a free finitely generated divisor class group $\mathrm{Cl}(X)$, which is transversally covered by cylinders, then the group $\mathrm{SAut}(\widehat{X})$ acts on the universal torsor $\widehat{X}$ transitively.

As a particular case, in Section 4 we study A-covered varieties, i.e. varieties covered by open subsets isomorphic to the affine space. Clearly, any A-covered variety is smooth and rational. We list wide classes of A-covered varieties including smooth complete toric or, more generally, spherical varieties, smooth rational projective surfaces, and some Fano threefolds. It is shown that the condition to be A-covered is preserved under passing to vector bundles and their projectivizations as well as to the blow up in a linear subvariety. In the appendix to this paper we prove that any smooth complete rational T-variety of complexity one is A-covered. This part uses the technique of polyhedral divisors from [1], [2].

In Section 5 we summarize our results on universal torsors and infinite transitivity. Theorem 3 claims that if X is an A-covered algebraic variety of dimension at least 2, then $\mathrm{SAut}(\widehat{X})$ acts on the universal torsor $\widehat{X}$ infinitely transitively. If the Cox ring $\mathcal{R}(X)$ is finitely generated, then the total coordinate space $\overline{X} := \mathrm{Spec}\, \mathcal{R}(X)$ is a factorial affine variety, the group $\mathrm{SAut}(\overline{X})$ acts on $\overline{X}$ with an open orbit O, and the action of $\mathrm{SAut}(\overline{X})$ on O is infinitely transitive, see Theorem 2. In particular, the Makar-Limanov invariant of $\overline{X}$ is trivial, see Corollary 1.

We work over an algebraically closed field $\mathbb{K}$ of characteristic zero.

1. Preliminaries on Cox rings and universal torsors

Let X be a normal algebraic variety with free finitely generated divisor class group $\mathrm{Cl}(X)$. Denote by $\mathrm{WDiv}(X)$ the group of Weil divisors on X and fix a subgroup $K \subseteq \mathrm{WDiv}(X)$ such that the canonical map $c\colon K \to \mathrm{Cl}(X)$ sending $D \in K$ to its class $[D] \in \mathrm{Cl}(X)$ is an isomorphism. We define the *Cox sheaf* associated to K to be

$$\mathcal{R} := \bigoplus_{[D]\in \mathrm{Cl}(X)} \mathcal{R}_{[D]}, \qquad \mathcal{R}_{[D]} := \mathcal{O}_X(D),$$

where $D \in K$ represents $[D] \in \mathrm{Cl}(X)$ and the multiplication in $\mathcal{R}$ is defined by multiplying homogeneous sections in the field of rational functions $\mathbb{K}(X)$. The sheaf $\mathcal{R}$ is a quasi-coherent sheaf of normal integral K-graded $\mathcal{O}_X$-algebras and, up to isomorphy, it does not depend on the choice of the subgroup $K \subseteq \mathrm{WDiv}(X)$, see [4, Construction I.4.1.1].

The *Cox ring* of X is the algebra of global sections

$$\mathcal{R}(X) := \bigoplus_{[D]\in \mathrm{Cl}(X)} \mathcal{R}_{[D]}(X),\ \mathcal{R}_{[D]}(X) := \Gamma(X, \mathcal{O}_X(D)).$$

Let us assume that X is a smooth variety with only constant invertible functions. Then the sheaf $\mathcal{R}$ is locally of finite type, and the relative spectrum $\mathrm{Spec}_X \mathcal{R}$ is a quasi-affine variety $\widehat{X}$, see [4, Corollary I.3.4.6]. We have $\Gamma(\widehat{X}, \mathcal{O}) \cong \mathcal{R}(X)$, and the ring $\mathcal{R}(X)$ is a unique factorization domain with only constant invertible elements, see [4, Proposition I.4.1.5]. Since the sheaf $\mathcal{R}$ is K-graded, the variety $\widehat{X}$ carries a natural action of the torus $H := \mathrm{Spec}\ \mathbb{K}[K]$. The projection $q\colon \widehat{X} \to X$ is called the *universal torsor* over the variety X. By [4, Remark I.3.2.7], the morphism $q\colon \widehat{X} \to X$ is a locally trivial H-principal bundle. In particular, the torus H acts on $\widehat{X}$ freely.

Lemma 1. *Let X be a normal variety. Assume that there is an open subset U on X which is isomorphic to the affine space $\mathbb{A}^n$. Then any invertible function on X is constant and the group $\mathrm{Cl}(X)$ is freely generated by classes $[D_1], \ldots, [D_k]$ of the prime divisors such that*

$$X \setminus U = D_1 \cup \ldots \cup D_k.$$

Proof. The restriction of an invertible function to U is constant, so the function is constant. Since U is factorial, any Weil divisor on X is linearly equivalent to a divisor whose support does not intersect U. This shows that the group $\mathrm{Cl}(X)$ is generated by $[D_1], \ldots, [D_k]$.

Assume that $a_1D_1 + \ldots + a_kD_k = \mathrm{div}(f)$ for some $f \in \mathbb{K}(X)$. Then f is a regular invertible function on U and thus f is a constant. This shows that the classes $[D_1], \ldots, [D_k]$ generate the group $\mathrm{Cl}(X)$ freely. $\square$

The Cox ring $\mathcal{R}(X)$ and the relative spectrum $q\colon \widehat{X} \to X$ can be defined and studied under weaker assumptions on the variety X, see [4, Chapter I]. But in this paper we are interested in smooth varieties with free finitely generated divisor class group.

Assume that the Cox ring $\mathcal{R}(X)$ is finitely generated. Then we may consider the *total coordinate space* $\overline{X} := \operatorname{Spec} \mathcal{R}(X)$. This is a factorial affine H-variety. By [4, Construction I.6.3.1], there is a natural open H-equivariant embedding $\widehat{X} \hookrightarrow \overline{X}$ such that the complement $\overline{X} \setminus \widehat{X}$ is of codimension at least two.

2. Special automorphisms and infinite transitivity

An action of a group G on a set A is said to be *m-transitive* if for every two tuples of pairwise distinct points $(a_1, \ldots, a_m)$

and $(a'_1, \ldots, a'_m)$ in A there exists $g \in G$ such that $g \cdot a_i = a'_i$ for $i = 1, \ldots, m$. An action which is m-transitive for all $m \in \mathbb{Z}_{>0}$ is called *infinitely transitive*.

Let Y be an algebraic variety. Consider a regular action $\mathbb{G}_a \times Y \to Y$ of the additive group $\mathbb{G}_a = (\mathbb{K}, +)$ of the ground field on Y. The image, say, L of $\mathbb{G}_a$ in the automorphism group $\mathrm{Aut}(Y)$ is a one-parameter unipotent subgroup. We let $\mathrm{SAut}(Y)$ denote the subgroup of $\mathrm{Aut}(Y)$ generated by all its one-parameter unipotent subgroups. Automorphisms from the group $\mathrm{SAut}(Y)$ are called *special*. In general, $\mathrm{SAut}(Y)$ is a normal subgroup of $\mathrm{Aut}(Y)$.

Denote by Y_{reg} the smooth locus of a variety Y. We say that a point $y \in Y_{\mathrm{reg}}$ is *flexible* if the tangent space $T_y Y$ is spanned by the tangent vectors to the orbits $L \cdot y$ over all one-parameter unipotent subgroups L in $\mathrm{Aut}(Y)$. The variety Y is *flexible* if every point $y \in Y_{\mathrm{reg}}$

is. Clearly, Y is flexible if one point of Y_{reg} is and the group $\mathrm{Aut}(Y)$ acts transitively on Y_{reg}. Many examples of flexible varieties are given in [6] and [5].

The following result is proven in [5, Theorem 0.1].

Theorem 1. *Let Y be an irreducible affine variety of dimension ≥ 2. Then the following conditions are equivalent.*

1. *The group $\mathrm{SAut}(Y)$ act transitively on Y_{reg}.*
2. *The group $\mathrm{SAut}(Y)$ act infinitely transitively on Y_{reg}.*
3. *The variety Y is flexible.*

A more general version of implication $1 \Rightarrow 2$ is given in [5, Theorem 2.2]. In this section we obtain an analog of this result for quasiaffine varieties, see Theorem 2 below.

Let Y be an algebraic variety. A regular action $\mathbb{G}_a \times Y \to Y$ defines a structure of a rational $\mathbb{G}_a$-algebra on $\Gamma(Y, \mathcal{O})$. The differential of this action is a locally nilpotent derivation D on $\Gamma(Y, \mathcal{O})$. Elements in $\operatorname{Ker} D$ are precisely the functions invariant under $\mathbb{G}_a$. The structure of a $\mathbb{G}_a$-module on $\Gamma(Y, \mathcal{O})$ can be reconstructed from D via exponential map.

Assume that Y is quasiaffine. Then regular functions separate points on Y. In particular, any automorphism of Y is uniquely defined by the induced automorphism of the algebra $\Gamma(Y, \mathcal{O})$. Hence a regular $\mathbb{G}_a$-action on Y can be reconstructed from the corresponding locally nilpotent derivation D. At the same time, if Y is not affine, then not every locally nilpotent derivation on $\Gamma(Y, \mathcal{O})$ gives rise to a regular $\mathbb{G}_a$-action on Y.

If D is a locally nilpotent derivation assigned to a $\mathbb{G}_a$-action on a quasiaffine variety Y and $f \in \operatorname{Ker} D$, then the derivation fD is locally nilpotent and it corresponds to a $\mathbb{G}_a$-action on Y with the same orbits on $Y \setminus \operatorname{div}(f)$, which fixes all points on the divisor $\operatorname{div}(f)$. The one-parameter subgroup of $\operatorname{SAut}(Y)$ defined by fD is called a *replica* of the subgroup given by D.

We say that a subgroup G of $\operatorname{Aut}(Y)$ is *algebraically generated* if it is generated as an abstract group by a family $\mathcal{G}$ of connected algebraic[1] subgroups of $\operatorname{Aut}(Y)$.

Proposition 1. [5, Proposition 1.5] *There are (not necessarily distinct) subgroups $H_1, \ldots, H_s \in \mathcal{G}$ such that*

$$G.x = (H_1 \cdot H_2 \cdot \ldots \cdot H_s) \cdot x \quad \forall x \in X. \tag{1}$$

A sequence $\mathcal{H} = (H_1, \ldots, H_s)$ satisfying condition (1) of Proposition 1 is called *complete*.

[1]not necessarily affine.

Let us say that a subgroup $G \subseteq \mathrm{SAut}(Y)$ is *saturated* if it is generated by one-parameter unipotent subgroups and there is a complete sequence $(H_1, \ldots, H_s)$ of one-parameter unipotent subgroups in G such that G contains all replicas of $H_1, \ldots, H_s$. In particular, $G = \mathrm{SAut}(X)$ is a saturated subgroup.

Theorem 2. *Let Y be an irreducible quasi-affine algebraic variety of dimension ≥ 2 and let $G \subseteq \mathrm{SAut}(Y)$ be a saturated subgroup, which acts with an open orbit $O \subseteq Y$. Then G acts on O infinitely transitively.*

Remark 1. Let H be a one-parameter unipotent subgroup of G. According to [26, Theorem 3.3], the field of rational invariants $\mathbb{K}(Y)^H$ is the field of fractions of the algebra $\mathbb{K}[Y]^H$ of regular invariants. Hence, by Rosenlicht's Theorem (see [26, Proposition 3.4]), regular invariants separate orbits on an H-invariant open dense subset $U(H)$ in Y. Furthermore, $U(H)$ can be chosen to be contained in O and consisting of 1-dimensional H-orbits.

For the remaining part of this section we fix the following notation. Let $H_1, \ldots, H_s$ be a complete sequence of one-parameter unipotent subgroups in G. We choose subsets $U(H_1), \ldots, U(H_s) \subseteq O$ as in Remark 1 and let

$$V = \bigcap_{k=1}^{s} U(H_k)\,.$$

In particular, V is open and dense in O. We say that a set of points $x_1, \ldots, x_m$ in Y is *regular*, if $x_1, \ldots, x_m \in V$ and $H_k \cdot x_i \neq H_k \cdot x_j$ for all $i, j = 1, \ldots, m$, $i \neq j$, and all $k = 1, \ldots, s$.

Remark 2. For any H_k, any 1-dimensional H_k-orbits $O_1, \ldots, O_r$ intersecting V and any $p = 1, \ldots, s$ we may choose a replica $H_{k,p}$

such that all O_q but O_p are pointwise $H_{k,p}$-fixed. To this end, we find H_k-invariant functions $f_{k,p,p'}$ such that $f_{k,p,p'}|_{O_p} = 1$, $f_{k,p,p'}|_{O_{p'}} = 0$. Then we take

$$H_{k,p} = \{ \exp(t(\prod_{p' \neq p} f_{k,p,p'})D_k) \ ; \ t \in \mathbb{K} \},$$

where D_k is a locally nilpotent derivation corresponding to H_k.

Lemma 2. *For every subset $x_1, \ldots, x_m \in O$ there exists an element $g \in G$ such that the set $g \cdot x_1, \ldots, g \cdot x_m$ is regular.*

Proof. For any x_i there holds $V \subset O = H_1 \cdots H_s \cdot x_i$. The condition $h_1 \cdots h_s \cdot x \in V$ is open and nonempty, hence we obtain an open subset $W \subset H_1 \times \ldots \times H_s$ such that $h_1 \cdots h_s \cdot x_i \in V$ for any $(h_1, \ldots, h_s) \in W$ and any x_i.

So we may suppose that $x_1, \ldots, x_m \in V$. Let N be the number of triples (i, j, k) such that $i \neq j$ and $H_k \cdot x_i = H_k \cdot x_j$. If $N = 0$, then the lemma is proved. Assume that $N \geq 1$ and fix such a triple (i, j, k).

There exists l such that $H_k \cdot x_i$ has at most finite intersection with H_l-orbits; otherwise $H_k \cdot x_i$ is invariant with respect to all $H_1, \ldots, H_s$, a contradiction with the condition $\dim O \geq 2$.

We claim that there is a one-parameter subgroup H in G such that

(2)
$H_k \cdot (h \cdot x_i) \neq H_k \cdot (h \cdot x_j)$ for all but finitely many

Let us take first $H = H_l$. Condition (2) is determined by a finite set of H_k-invariant functions. So, either it holds or $H_k \cdot (h \cdot x_i) = H_k \cdot (h \cdot x_j)$ for all $h \in H$.

Assume that $H_l \cdot x_i \neq H_l \cdot x_j$. By Remark 2 there exists a replica H_l' such that $H_l' \cdot x_i = x_i$, but $H_l' \cdot x_j = H_l \cdot x_j$. We take $H = H_l'$, and condition (2) is fulfilled.

Assume now the contrary. Then there exists $h_l \in H_l$ such that $h_l \cdot x_i = x_j$. Then the set $\{h_l^n \cdot x_i \mid n \in \mathbb{Z}_{>0}\}$ has finite intersection with any H_k-orbit, and $h_l^n \cdot x_j = h_l^{n+1} \cdot x_i$ lie in different H_k-orbits for an infinite set of $n \in \mathbb{Z}_{>0}$. Therefore, this holds for an open subset of H_l, and condition (2) is again fulfilled.

Finally, the following conditions are open and nonempty on H:

- (C1) $h \cdot x_1, \ldots, h \cdot x_m \in V$;
- (C2) if $H_p \cdot x_{i'} \neq H_p \cdot x_{j'}$ for some p and $i' \neq j'$, then $H_p \cdot (h \cdot x_{i'}) \neq H_p \cdot (h \cdot x_{j'})$.

Hence there exists $h \in H$ satisfying (C1), (C2), and condition (2). We conclude that for the set $(h \cdot x_1, \ldots, h \cdot x_m)$ the value of N is smaller, and proceed by induction. □

Lemma 3. *Let $x_1, \ldots, x_m$ be a regular set and $G(x_1, \ldots, x_{m-1})$ be the intersection of the stabilizers of the points $x_1, \ldots, x_{m-1}$ in G. Then* *the orbit* $G(x_1, \dots, x_{m-1}) \cdot x_m$ *contains an open subset in* O.

Proof. We claim that there is a nonempty open subset $U \subseteq H_1 \times \dots \times H_s$ such that for every $(h_1, \dots, h_s) \in U$ we have

$$h_1 \dots h_s \cdot x_m = g \cdot x_m \quad \text{for some } g \in G(x_1, \dots, x_{m-1}$$

Indeed, let Z be the union of orbits $H_k \cdot x_i$, $k = 1, \dots, s$, $i = 1, \dots, m-1$. The set $V \setminus Z$ is open and contains x_m. Let U be the set of all $(h_1, \dots, h_s)$ such that $h_r \dots h_s \cdot x_m \in V \setminus Z$ for any $r = 1, \dots, s$. Then U is open and nonempty. Let us show that for any $(h_1, \dots, h_s) \in U$ and any $r = 1, \dots, s$ the point $h_r \dots h_s \cdot x_m$ is in the orbit $G(x_1, \dots, x_{m-1}) \cdot x_m$. Assume that $h_{r+1} \dots h_s \cdot x_m \in G(x_1, \dots, x_{m-1}) \cdot x_m$. By Remark 2, there is a replica H_r' of the subgroup H_r which fixes $x_1, \dots, x_{m-1}$ and such that the orbits

$$H_r \cdot (h_{r+1} \dots h_s \cdot x_m) \quad \text{and} \quad H'_r \cdot (h_{r+1} \dots h_s \cdot x_m)$$

coincide. Then H'_r is contained in $G(x_1, \dots, x_{m-1})$ and the point $h_r h_{r+1} \dots h_s \cdot x_m$ is in the orbit $G(x_1, \dots, x_{m-1}) \cdot x_m$ for any $h_r \in H_r$. The claim is proved.

Now the image of the dominant morphism

$$U \to O, \quad (h_1, \dots, h_s) \mapsto h_1 \dots h_s \cdot x_m$$

contains an open subset in O. □

Proof of Theorem 2. Let $(x_1, \dots, x_m)$ and $(y_1, \dots, y_m)$ be two sets of pairwise distinct points in O. We have to show that there is an element $g \in G$ such that $g \cdot x_1 = y_1, \dots, g \cdot x_m = y_m$.

We argue by induction on m. If $m = 1$, then the claim is obvious. If $m > 1$, then by inductive hypothesis there exists $g' \in G$ such that $g' \cdot x_1 = y_1, \dots, g' \cdot x_{m-1} = y_{m-1}$. If $g' \cdot x_m = y_m$, the assertion is proved. Assume that $g' \cdot x_m \neq y_m$. By Lemma 2, there exists $g'' \in G$ such that the set

$$g'' \cdot y_1, \ldots, g'' \cdot y_{m-1}, g'' \cdot y_m, g''g' \cdot x_m$$

is regular. Lemma 3 implies that the orbits

$G(g''\cdot y_1, \ldots, g''\cdot y_{m-1})\cdot(g''\cdot y_m)$ and $G(g''\cdot y_1, \ldots$

intersect, so there is $g''' \in G(g'' \cdot y_1, \ldots, g'' \cdot y_{m-1})$ such that $g'''g''g'x_m = g''y_m$. Then the element $g = (g'')^{-1}g'''g''g'$ is as desired. □

3. Cylinders and $\mathbb{G}_a$-actions

The following definition is taken from [21], see also [22].

Definition 1. Let X be an algebraic variety and U be an open subset of X. We say that U is a *cylinder* if $U \cong Z \times \mathbb{A}^1$, where Z is an irreducible affine variety with $\mathrm{Cl}(Z) = 0$.

Proposition 2. *Let X be a smooth algebraic variety with a free finitely generated divisor* *class group* $\mathrm{Cl}(X)$, $q\colon \widehat{X} \to X$ *be the universal torsor, and* $U \cong Z \times \mathbb{A}^1$ *be a cylinder in* X. *Then there is an action* $\mathbb{G}_a \times \widehat{X} \to \widehat{X}$ *such that*

(i) the set of $\mathbb{G}_a$*-fixed points is* $\widehat{X} \setminus q^{-1}(U)$*;*
(ii) for any point $y \in q^{-1}(U)$ *we have* $q(L \cdot y) = \{z\} \times \mathbb{A}^1$ *for some* $z \in Z$*, where* L *is the image of* $\mathbb{G}_a$ *in* $\mathrm{Aut}(\widehat{X})$.

Proof. Since $\mathrm{Cl}(U) \cong \mathrm{Cl}(Z) = 0$, we have an isomorphism $q^{-1}(U) \cong Z \times \mathbb{A}^1 \times H$ compatible with the projection q, see [4, Remark I.3.2.7]. Thus the subset $q^{-1}(U)$ admits a $\mathbb{G}_a$-action

$$a \cdot (z, t, h) = (z, t+a, h), \quad z \in Z,\ t \in \mathbb{A}^1,\ h \in H,$$

with property (ii). Denote by D the locally nilpotent derivation on $\Gamma(U, \mathcal{O})$ corresponding to this action.

Our aim is to extend the action to $\widehat{X}$. Since the open subset $q^{-1}(U)$ is affine, its complement $\widehat{X} \setminus q^{-1}(U)$ is a divisor Δ in $\widehat{X}$. We can find a function $f \in \Gamma(\widehat{X}, \mathcal{O})$ such that

$\Delta = \mathrm{div}(f)$. In particular,

$$\Gamma(q^{-1}(U), \mathcal{O}) = \Gamma(\widehat{X}, \mathcal{O})[1/f].$$

Since f has no zero on any $\mathbb{G}_a$-orbit on $q^{-1}(U)$, it is constant along orbits, and f lies in $\mathrm{Ker}\, D$.

Lemma 4. *Let Y be an irreducible quasiaffine variety,*

$$Y = \bigcup_{i=1}^{s} Y_{g_i}, \qquad g_i \in \Gamma(Y, \mathcal{O}),$$

be an open covering by principle affine subsets, and let

$$\Gamma(Y_{g_i}, \mathcal{O}) = \mathbb{K}[c_{i1}, \ldots, c_{ir_i}][1/g_i]$$

for some $c_{ij} \in \Gamma(Y, \mathcal{O})$. Consider a finitely generated subalgebra C in $\Gamma(Y, \mathcal{O})$ containing all the functions g_i and c_{ij}. Then the natural morphism $Y \to \mathrm{Spec}\ C$ is an open embedding.

Proof. Notice that $\Gamma(Y_{g_i}, \mathcal{O}) = \Gamma(Y, \mathcal{O})[1/g_i] = C[1/g_i]$. This shows that the morphism

$Y \to \operatorname{Spec} C$ induces isomorphisms $Y_{g_i} \cong (\operatorname{Spec} C)_{g_i}$. $\square$

Let $Y = \widehat{X}$ and $\widehat{X} \hookrightarrow \operatorname{Spec} C$ be an affine embedding as in Lemma 4 with $f \in C$. A finite generating set of the algebra C is contained in a finite dimensional D-invariant subspace W of $\Gamma(q^{-1}(U), \mathcal{O})$. Replacing D with $f^m D$ we may assume that W is contained in $\Gamma(\widehat{X}, \mathcal{O})$. We enlarge C and assume that it is generated by W. Then C is an $(f^m D)$-invariant finitely generated subalgebra in $\Gamma(\widehat{X}, \mathcal{O})$ and we have an open embedding $\widehat{X} \hookrightarrow \operatorname{Spec} C =: \widetilde{X}$.

Replacing $f^m D$ with $D' := f^{m+1} D$, we obtain a locally nilpotent derivation D' on C such that $D'(C)$ is contained in fC. The corresponding $\mathbb{G}_a$-action on $\widetilde{X}$ fixes all points on $\operatorname{div}(f)$ and has the same orbits on $q^{-1}(U)$. Hence the subset $\widehat{X} \subseteq \widetilde{X}$ is $\mathbb{G}_a$-invariant and the restriction of the action to $\widetilde{X}$ has the desired properties. The proof of Proposition 2 is completed. $\square$

Remark 3. Under the assumption that the algebra $\Gamma(\widetilde{X}, \mathcal{O})$ is finitely generated the proof of Proposition 2 is much simpler.

The following definitions appeared in [25].

Definition 2. Let X be a variety and $U \cong Z \times \mathbb{A}^1$ be a cylinder in X. A subset W of X is said to be *U-invariant* if $W \cap U = p_1^{-1}(p_1(W \cap U))$, where $p_1 \colon U \to Z$ is the projection to the first factor. In other words, every $\mathbb{A}^1$-fiber of the cylinder is either contained in W or does not meet W.

Definition 3. We say that a variety X is *transversally covered* by cylinders U_i, $i = 1, \ldots, s$, if $X = \bigcup_{i=1}^{s} U_i$ and there is no proper subset $W \subset X$ invariant under all U_i.

Proposition 3. *Let X be a smooth algebraic variety with a free finitely generated divisor class group $\mathrm{Cl}(X)$ and $q\colon \widehat{X} \to X$ be the universal torsor. Assume that X is transversally covered by cylinders. Then the group $\mathrm{SAut}(\widehat{X})$ acts on $\widehat{X}$ transitively.*

Proof. Consider a $\mathbb{G}_a$-action on $\widehat{X}$ associated with the cylinder U_i as in Proposition 2. Let L_i be the corresponding $\mathbb{G}_a$-subgroup in $\mathrm{SAut}(\widehat{X})$ and G be the subgroup of $\mathrm{SAut}(\widehat{X})$ generated by all the L_i. By construction, the subgroups L_i and thus the group G commute with the torus H.

Let S be a G-orbit on $\widehat{X}$. By Proposition 2, the projection $q(S)$ is invariant under all the cylinders U_i, and thus $q(S)$ coincides with X. Let H_S be the stabilizer of the subset S in H. Then the map $H \times S \to \widehat{X}$, $(h, x) \mapsto hx$, is surjective and its image is isomorphic to $(H/H_S) \times S$. Since H/H_S is a torus and the variety $\widehat{X}$ has only constant invertible functions, we conclude that $H_S = H$ and thus $S = \widehat{X}$. This shows that G, and hence $\mathrm{SAut}(\widehat{X})$, acts on $\widehat{X}$ transitively. □

4. A-covered varieties

The affine space $\mathbb{A}^n$ admits n coordinate cylinder structures $\mathbb{A}^{n-1} \times \mathbb{A}^1$, and the covering of $\mathbb{A}^n$ by these cylinders is transversal. This elementary observation motivates the following definition.

Definition 4. An irreducible algebraic variety X is said to be *A-covered* if there is an open covering $X = U_1 \cup \ldots \cup U_r$, where every chart U_i is isomorphic to the affine space $\mathbb{A}^n$.

A choice of such a covering together with isomorphisms $U_i \cong \mathbb{A}^n$ is called an *A-atlas* of X. A subvariety Z of an A-covered variety X is called *linear* with respect to an A-atlas, if it is linear in all charts, i.e. $Z \cap U_i$ is a linear subspace in $U_i \cong \mathbb{A}^n$. Any A-covered variety is rational, smooth, and by Lemma 1 the group $\mathrm{Pic}(X) = \mathrm{Cl}(X)$ is finitely generated and free.

Clearly, the projective space $\mathbb{P}^n$ is A-covered. This fact can be generalized in several ways.

1) Every smooth complete toric variety X is A-covered.
2) Every smooth rational complete variety with a torus action of complexity one is A-covered; see the appendix to this paper.
3) Let G be a semisimple algebraic group and be P a parabolic subgroup of G. Then the flag variety G/P is A-covered. Indeed, a maximal unipotent subgroup N of G acts on G/P with an open orbit U isomorphic to an affine space. Since G acts

on G/P transitively, we obtain the desired covering.

4) More generally, every complete smooth spherical variety is A-covered, see [10, Corollary 1.5].
5) The Fano threefolds $\mathbb{P}^3$, Q, V_5 and an element of the family V_{22} are known to be A-covered. Moreover, there are no other types of A-covered threefolds of Picard number 1 by [15]. In particular, the Fano threefolds V_{12}, V_{16}, V_{18} and V_4 from Iskovskikh's classification [20] are rational but not A-covered.
6) The product of two A-covered varieties is again A-covered.
7) More generally, every vector bundle over $\mathbb{A}^n$ trivializes, and total spaces of vector bundles over A-covered varieties are A-covered. The same holds for their projectivizations.

8) If a variety X is A-covered and X' is a blow up of X at some point $p \in X$, then X' is A-covered.
9) In particular, all smooth projective rational surfaces are obtained either from $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or from the Hirzebruch surfaces F_n by a sequence of blow ups of points, and thus they are A-covered.
10) We may generalize the blow up example as follows. The blow up of X in a linear subvariety Z is A-covered. Moreover, the strict transforms of linear subvarieties, which either contain Z or do not intersect with it, are linear again (with the choice of an appropriate A-atlas). Hence, we may iterate this procedure.

Proof of statement 10). We consider one chart U of the covering on X. We may assume, that we blow up $\mathbb{A}^n = U$ in the linear subspace given by $x_1 = \ldots = x_k = 0$.

By definition, the blow up X' is given in the product $\mathbb{A}^n \times \mathbb{P}^{k-1}$ by equations $x_i z_j = x_j z_i$, where $1 \leq i, j \leq k$. If the homogeneous coordinate z_j equals 1 for some $j = 1, \ldots, k$, then $x_i = x_j z_i$, and we are in the open chart V_j with independent coordinates x_j, x_s with $s > k$, and z_i, $i \neq j$. So the variety X' is covered by k such charts.

Let L be a linear subspace in U containing $[x_1 = \ldots = x_k = 0]$ and given by linear equations $f_i(x_1, \ldots, x_k) = 0$. The strict transform of L is given in V_j by the equations $f_i(z_1, \ldots, z_{j-1}, 1, z_{j+1}, \ldots, z_k) = 0$. After a change of variables $x_j \mapsto x_j - 1$ these equations become linear.

Finally, if a linear subvariety Z' does not meet the linear subvariety Z, then Z' does not intersect charts of our atlas that intersect Z, and the assertion follows. $\square$

Example 1. Consider the quadric threefold Q. Choose two points and a conic passing through them. Then these are linear subvarieties of Q with respect to an appropriate atlas. Hence, the iterated blow up in the points, first, and then in the strict transform of the conic is A-covered.

We may use the above observations to take a closer look at Fano threefolds.

Proposition 4. *In the classification of Iskovskikh* [20] *and Mori-Mukai* [24] *we have the following (possibly non-complete) list of A-covered Fano threefolds:*

a) $\mathbb{P}^3$, Q, V_5, *(at least) one element* V_{22}' *of the family* V_{22}*;*
b) 2.33-2.36, 3.26-3.31, 4.9-4.11, 5.2, 5.3;
c) 2.29, 2.30, 2.31, 2.32, 3.8, 3.18-3.23, 3.24, 4.4, 4.7, 4.8, (at least) one element of the families 2.24, 3.8 and 3.10 respectively;

d) 5.3-5.8;
e) (at least) one element of the family 2.26.

Proof. List a) is the same as 5). List b) are exactly the toric Fano threefolds. The varieties in c) admit a 2-torus action. This can be seen more or less directly from the description given in [24]. For some of them we get alternative proofs of the A-coveredness by 3), 7) and 10). The varieties in d) are products of del Pezzo surfaces (which are rational) and $\mathbb{P}^1$. The variety in e) is obtained from V_5 by blow up in linear subvariety as explained in 10). □

5. Main results

The following theorem summarizes our results on universal torsors and infinite transitivity.

Theorem 3. *Let X be an A-covered algebraic variety of dimension at least 2 and*

$q\colon \widehat{X} \to X$ be the universal torsor. Then the group $\mathrm{SAut}(\widehat{X})$ *acts on the quasiaffine variety* $\widehat{X}$ *infinitely transitively.*

Proof. If X is covered by m open charts isomorphic to $\mathbb{A}^n$, and every chart is equipped with n transversal cylinder structures, then the covering of X by these mn cylinders is transversal. By Proposition 3, the group $\mathrm{SAut}(\widehat{X})$ acts on $\widehat{X}$ transitively. Theorem 2 yields that the action is infinitely transitive. $\square$

Theorem 3 provides many examples of quasiaffine varieties with rich symmetries. In particular, if X is a del Pezzo surface, a description of the universal torsor $q\colon \widehat{X} \to X$ may be found in [7], [27], [28]. It follows from Theorem 3 that the group $\mathrm{SAut}(\widehat{X})$ acts on $\widehat{X}$ infinitely transitively.

If X is the blow up of nine points in general position on $\mathbb{P}^2$, that it is well known that the Cox ring $\mathcal{R}(X)$ is not finitely generated, and thus $\widehat{X}$ is a quasiaffine variety with a non-finitely generated algebra of regular functions $\Gamma(\widehat{X}, \mathcal{O})$. Theorem 3 works in this case as well.

Theorem 4. *Let X be an A-covered algebraic variety of dimension at least 2. Assume that the Cox ring $\mathcal{R}(X)$ is finitely generated. Then the total coordinate space $\overline{X} := \operatorname{Spec} \mathcal{R}(X)$ is a factorial affine variety, the group $\mathrm{SAut}(\overline{X})$ acts on $\overline{X}$ with an open orbit O, and the action of $\mathrm{SAut}(\overline{X})$ on O is infinitely transitive.*

Proof. Lemma 1 shows that the group $\mathrm{Cl}(X)$ is finitely generated and free, hence the ring $\mathcal{R}(X)$ is a unique factorization domain, see [4, Proposition I.4.1.5]. Since

$$\Gamma(\overline{X}, \mathcal{O}) = \mathcal{R}(X) \cong \Gamma(\widehat{X}, \mathcal{O}),$$

any $\mathbb{G}_a$-action on $\widehat{X}$ extends to $\overline{X}$. We conclude that $\widehat{X}$ is contained in one $\mathrm{SAut}(\overline{X})$-orbit O on $\overline{X}$, the action of $\mathrm{SAut}(\overline{X})$ on O is infinitely transitive, and by [5, Proposition 1.3] the orbit O is open in $\overline{X}$. □

Recall from [14] that the *Makar-Limanov invariant* $\mathrm{ML}(Y)$ of an affine variety Y is the intersection of the kernels of all locally nilpotent derivations on $\Gamma(Y, \mathcal{O})$. In other words $\mathrm{ML}(Y)$ is the subalgebra of all $\mathrm{SAut}(Y)$-invariants in $\Gamma(Y, \mathcal{O})$. Similarly to as in [23] the *field Makar-Limanov invariant* $\mathrm{FML}(Y)$ is the subfield of $\mathbb{K}(Y)$ which consists of all rational $\mathrm{SAut}(Y)$-invariants. If the field Makar-Limanov invariant is trivial, that is, if $\mathrm{FML}(Y) = \mathbb{K}$, then so is $\mathrm{ML}(Y)$, but the converse is not true in general.

Corollary 1. *Under the assumptions of Theorem 4 the field Makar-Limanov invariant $\mathrm{FML}(\overline{X})$ is trivial.*

Proof. By Theorem 4, the group $\mathrm{SAut}(\overline{X})$ acts on $\overline{X}$ with an open orbit. So any rational $\mathrm{SAut}(\overline{X})$-invariant is constant. □

Appendix: Rational T-varieties of complexity one

By a T-variety we mean a normal variety equipped with an effective action of an algebraic torus T. The difference of dimensions $\dim X - \dim T$ is called the *complexity* of a T-variety. Hence, toric varieties are T-varieties of complexity zero. For the case of complexity one we want to prove the following theorem.

Theorem 5. *Any smooth complete rational T-variety of complexity one is A-covered.*

Due to [1] T-varieties can be described and studied in the language of polyhedral divisors. As for ordinary divisors we have to associate certain coefficients to codimension one subvarieties of some base variety Y. But in our case the coefficients are not integers or real numbers, but polyhedra in some vector space.

Our general references are [1, 2, 3], but we restrict ourself to the case of rational T-varieties of complexity one. In the language of polyhedral divisors this is equivalent to the fact that our divisors live on $\mathbb{P}^1$. This allows us to simplify some definitions.

The affine case. We consider a lattice M of rank n, the dual lattice $N = \mathrm{Hom}(M, \mathbb{Z})$, and the vector space $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $T = N \otimes_{\mathbb{Z}} \mathbb{K}^*$ be the algebraic torus of dimension n with character lattice M.

Every polyhedron $\Delta \subset N_{\mathbb{Q}}$ has a Minkowski decomposition $\Delta = P + \sigma$, where P is a (compact) polytope and σ is a polyhedral cone.

We call σ the *tail cone* of Δ and denote it by $\operatorname{tail}(\Delta)$.

A *polyhedral divisor* on $\mathbb{P}^1$ over N is a formal sum

$$\mathcal{D} = \sum_{y \in \mathbb{P}^1} \mathcal{D}_y \cdot y,$$

where $\mathcal{D}_y$ are polyhedra with common pointed tail cone σ and only finitely many coefficients differ from σ itself. Note that we allow empty coefficients. We set $Y_{\mathcal{D}} = \{y \in \mathbb{P}^1 \mid \mathcal{D}_y \neq \emptyset\}$.

We call $\mathcal{D}$ a *proper* polyhedral divisor or a *p-divisor* for short, if

$$\deg \mathcal{D} := \sum_{y \in \mathbb{P}^1} \mathcal{D}_y \subsetneq \sigma. \tag{3}$$

For every $u \in M$ we may evaluate the p-divisor and obtain a divisor on $Y_{\mathcal{D}}$ with coefficients in $\mathbb{Q}$, namely $\mathcal{D}(u) := \sum_y \min\langle u, \mathcal{D}_y \rangle \cdot y$.

Now a p-divisor gives rise to a finitely generated M-graded $\mathbb{K}$-algebra

$$\begin{aligned} A(\mathcal{D}) &= \bigoplus_{u \in \sigma^\vee \cap M} \Gamma(Y_{\mathcal{D}}, \mathcal{O}(\mathcal{D}(u))) \\ &= \bigoplus_{u \in \sigma^\vee \cap M} \{f \in \mathbb{K}(Y) \mid \operatorname{ord}_y f \geq -\inf\langle u, \mathcal{D}_y \rangle\}. \end{aligned}$$

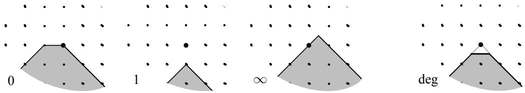
We obtain a normal rational affine variety $\mathfrak{X}(\mathcal{D}) := \operatorname{Spec} A(\mathcal{D})$ of dimension $n+1$ with a T-action induced by the M-grading. Every such variety arises this way [1, Theorems 3.1, 3.4].

Proposition 5. [3, Theorem 4] *Let $\mathcal{D}$ be a p-divisor.*

1. *For every $y \in \mathbb{P}^1$ choose a point $v_y \in N$, such that only finitely many of them differ from 0 and $\sum_y v_y = 0$. Then $\mathcal{D}$ and $\mathcal{D}' = \sum_{y \in \mathbb{P}^1} (\mathcal{D}_y + v_y) \cdot y$ give rise to isomorphic varieties.*

2. *If* $\varphi \in \mathrm{Aut}(\mathbb{P}^1)$ *then* $\mathcal{D}$ *and* $\varphi^*\mathcal{D} := \sum_y \mathcal{D}_{\varphi(y)} \cdot y$ *give rise to isomorphic varieties.*

Example 2. We consider a polyhedral divisor $\mathcal{D}$ on $\mathbb{P}^1$ having three polyhedral coefficients $\mathcal{D}_0$, $\mathcal{D}_1$ and $\mathcal{D}_\infty$ given in the first three images. All other coefficients equal the tail cone, which is spanned by $(-1,-1)$ and $(1,-1)$. The polyhedral divisor is proper, since the degree polyhedron is a proper subset of the tail cone as the last picture shows.



Example 3. [18, Remark 1.8.] Let us fix two points $y_0, y_\infty \in \mathbb{P}^1$. For $y \in \mathbb{P}^1 \setminus \{y_0, y_\infty\}$ we consider lattice points $v_y \in N$ such that only finitely many of them are different from 0. We denote the sum $\sum_{y \neq y_0, y_\infty} v_y$ by v and choose $w_0, w_\infty \in N$ with $w_0 + w_\infty = v$.

A polyhedral divisor of the form

$$\mathcal{D}_0 \cdot y_0 \;+\; \mathcal{D}_\infty \cdot y_\infty \;+\; \sum_y (v_y + \sigma) \cdot y \tag{4}$$

on $\mathbb{P}^1$ corresponds to the affine toric variety of the cone in $N_\mathbb{Q} \oplus \mathbb{Q}$ defined by $\operatorname{cone}(w_0 + \mathcal{D}_0, w_\infty + \mathcal{D}_\infty) := \mathbb{Q}_{\geq 0} \cdot ((w_0 + \mathcal{D}_0) \times \{1\} \cup \sigma \times \{0\} \cup (w_\infty + \mathcal{D}_\infty) \times \{-1\})$ together with the subtorus action given by the lattice embedding $N \hookrightarrow N \oplus \mathbb{Z}$. Here, we allow $\mathcal{D}_0 = \emptyset$ or $\mathcal{D}_\infty = \emptyset$. Different choices of w_0 and w_∞ lead to cones which can be transformed into each other by a lattice automorphism of $N \times \mathbb{Z}$. Hence, the corresponding toric varieties are isomorphic and the above statement makes indeed sense. If the affine toric variety is assumed to be smooth, the cone has to be regular. If $\mathcal{D}_0$ or $\mathcal{D}_\infty$ has dimension n, then the constructed cone has dimension $n + 1$ and the variety is an affine space.

We may look at the concrete polyhedral divisor from Example 2. It has the desired form: only $\mathcal{D}_0$ and $\mathcal{D}_\infty$ are not lattice translates of the tail cone. The coefficient $\mathcal{D}_1$ is just the tail cone translated by $(0, -1)$. Hence, $v = (0, -1)$ holds. We may choose $w_0 = v$ and $w_\infty = 0$. Now $\operatorname{cone}(w_0 + \mathcal{D}_0, w_\infty + \mathcal{D}_\infty)$ is spanned by the rays $\mathbb{Q}_{\geq 0} \cdot (-1, -1, 1)$, $\mathbb{Q}_{\geq 0} \cdot (0, -1, 1)$ and $\mathbb{Q}_{\geq 0} \cdot (1, 1, -2)$. Since the ray generators form a basis in $\mathbb{Z}^3$, the corresponding T-varieties is an affine space with a 2-torus action.

It is not hard to exhibit in general the extremal rays of the cone constructed in Example 3.

Lemma 5. *There are three types of extremal rays in* $\operatorname{cone}(w_0 + \mathcal{D}_0, w_\infty + \mathcal{D}_\infty)$*:*

1. $\rho \times \{0\}$ *for every* $\rho \in \sigma(1)$*, where* $\deg \mathcal{D} \cap \rho = (w_0 + w_\infty + \mathcal{D}_0 + \mathcal{D}_\infty) \cap \rho = \emptyset$*;*

2. $\mathbb{Q}_{\geq 0} \cdot (w_0 + v, 1)$, *where* $v \in \mathcal{D}_0$ *is a vertex;*
3. $\mathbb{Q}_{\geq 0} \cdot (w_\infty + v, -1)$, *where* $v \in \mathcal{D}_\infty$ *is a vertex.*

Proposition 6. [30, Proposition 3.1 and Theorem 3.3.] *Let $\mathcal{D}$ be a p-divisor on $\mathbb{P}^1$. Then $\mathfrak{X}(\mathcal{D})$ is smooth if and only if*

1. *either* $\deg \mathcal{D} \neq \emptyset$, *$\mathcal{D}$ is of the form (4), and hence $\mathfrak{X}(\mathcal{D})$ is an affine space, or*
2. $\deg \mathcal{D} = \emptyset$ *and* $\mathrm{cone}(\mathcal{D}_y) := \mathrm{cone}(\mathcal{D}_y, \emptyset)$ *is regular for every $y \in \mathbb{P}^1$.*

Note that polyhedral divisors of the second type do not necessarily correspond to affine spaces. This is only the case if at most two coefficients are not lattice translates of the tail cone, see Example 3.

As a consequence of Lemma 5 and Proposition 6 we easily obtain that for two special cases all coefficients of $\mathcal{D}$ have to be translated cones in order to obtain a smooth affine variety.

Corollary 2. *Assume that $\mathfrak{X}(\mathcal{D})$ is smooth. If $\mathcal{D}$ has a tail cone σ of maximal dimension and $\deg \mathcal{D} \cap \tau = \emptyset$ for some facet $\tau \prec \sigma$, then all the coefficients are translates of σ and all but two are even lattice translates.*

Corollary 3. *If $\deg \mathcal{D} = \emptyset$ and $\mathfrak{X}(\mathcal{D})$ is smooth, then the tail cone σ has to be regular. Moreover, if σ is maximal, then $\mathcal{D}_y$ is either empty or a lattice translate of σ for every $y \in \mathbb{P}^1$.*

Complete case and affine coverings. Given two p-divisors $\mathcal{D}'$ and $\mathcal{D}$ such that $\mathcal{D}'_y \subset \mathcal{D}_y$ we obtain an inclusion of algebras $A(\mathcal{D}) \subset A(\mathcal{D}')$ and a dominant morphism of affine varieties $\mathfrak{X}(\mathcal{D}') \to \mathfrak{X}(\mathcal{D})$. If the latter is an open inclusion we write $\mathcal{D}' \prec \mathcal{D}$. For two p-divisors $\mathcal{D}$ and $\mathcal{D}'$ we define their *intersection* as $\mathcal{D}\cap\mathcal{D}' := \sum_y(\mathcal{D}_y\cap\mathcal{D}'_y)\cdot y$. This is again a p-divisor. If $\mathcal{D} \succ \mathcal{D}\cap\mathcal{D}' \prec \mathcal{D}'$ holds we may glue $\mathfrak{X}(\mathcal{D})$ and $\mathfrak{X}(\mathcal{D}')$ via the induced open inclusions of $\mathfrak{X}(\mathcal{D} \cap \mathcal{D}')$. More generally, from a finite set $\mathcal{S}$ of p-divisors fulfilling pairwise the condition $\mathcal{D} \succ \mathcal{D} \cap \mathcal{D}' \prec \mathcal{D}'$ we obtain a scheme by gluing the affine pieces via identification of common open affine subsets, see [2, Theorem 5.3].

The other way around, for every T-variety X we may consider a T-invariant covering by affine varieties $\mathfrak{X}(\mathcal{D})$ for p-divisors $\mathcal{D}$ from a finite set $\mathcal{S}$ as above [2, Theorem 5.6]. In the case of rational T-varieties of complexity one the relation $\mathcal{D}' \prec \mathcal{D}$ can be characterized explicitly: $\mathcal{D}' \prec \mathcal{D}$ if and only if $\mathcal{D}'_y$ is a face of $\mathcal{D}_y$ for every $y \in \mathbb{P}^1$ and $\deg \mathcal{D}' = \deg \mathcal{D} \cap \operatorname{tail} \mathcal{D}'$, see [19, Proposition 1.1]. Together with [2, Remark 7.4(iv)] this implies that a set $\mathcal{S}$ as above satisfies the following compatibility conditions.

Slice rule: The *slices* $\mathcal{S}_y = \{\mathcal{D}_y \mid \mathcal{D} \in \mathcal{S}\}$ are complete polyhedral subdivisions of $N_\mathbb{Q}$, i.e. they cover $N_\mathbb{Q}$ and the intersection of every two polyhedra is a face of both of them.

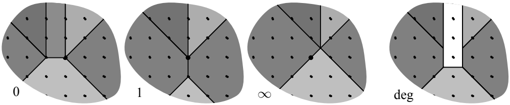
Degree rule: For $\tau = (\operatorname{tail}\mathcal{D}) \cap (\operatorname{tail}\mathcal{D}')$ one has $\tau \cap (\deg\mathcal{D}) = \tau \cap (\deg\mathcal{D}')$.

Note that $\operatorname{tail}\mathcal{S} := \{\operatorname{tail}\mathcal{D} \mid \mathcal{D} \in \mathcal{S}\}$ generates a fan and all but finitely many *slices* $\mathcal{S}_y$ just equal $\operatorname{tail}\mathcal{S}$. Consider a maximal tail cone σ in $\operatorname{tail}\mathcal{S}$. Then for every y there is a unique polyhedron $\mathcal{S}_y(\sigma)$ in $\mathcal{S}_y$ having this tail.

A maximal cone $\sigma \in \operatorname{tail}\mathcal{S}$ is called *marked* if the corresponding polyhedral divisor $\mathcal{D}$ with $\sigma = \operatorname{tail}\mathcal{D}$ fulfills $\deg\mathcal{D} \neq \emptyset$. We denote the set of all marked cones by $\operatorname{tail}^m(\mathcal{S}) \subset \operatorname{tail}(\mathcal{S})$.

In general, there are many torus invariant affine coverings of X. But by [19, Proposition 1.6] every rational complete T-variety of complexity one is uniquely determined by the slices $\mathcal{S}_y$ and the markings in $\operatorname{tail} \mathcal{S}$. Hence, another set $\mathcal{S}'$ of p-divisors with $\mathcal{S}_y = \mathcal{S}'_y$ for all $y \in \mathbb{P}^1$ and $\operatorname{tail}^m(\mathcal{S}) = \operatorname{tail}^m(\mathcal{S}')$ corresponds to another invariant affine covering of the same variety.

Example 4. We consider the blow up of the quadric threefold Q in one point. This is a T-variety of complexity one and the slices of a set $\mathcal{S}$ look as in the first three pictures. The last picture shows the degrees of the elements of $\mathcal{S}$.



>From now on we assume that X is a rational complete *smooth* T-variety of complexity

one and we consider an affine covering given by the p-divisors in $\mathcal{S}$. By Proposition 6, we have

Lemma 6. *Given a maximal cone σ in* $\operatorname{tail}\mathcal{S}$, *there are two possible cases:*

1. *σ is marked and all but two coefficients of $\mathcal{S}_y(\sigma)$ are lattice translates of σ, or*
2. *σ is not marked; then it has to be regular and $\mathcal{S}_y(\sigma)$ has to be a lattice translate of σ for every $y \in \mathbb{P}^1$.*

In the slices $\mathcal{S}_y$ there might occur maximal polyhedra with non-maximal tail cones, as $\sqcup := \overline{(-1,0)(0,0)} + \mathbb{Q}_{\geq 0} \cdot (0,1)$ in the first slice in Example 4. Here, Lemma 6 does not apply. Instead we need the following crucial fact.

Proposition 7. *Let P be a maximal polyhedron with non-maximal tail in $\mathcal{S}_z$ for some $z \in \mathbb{P}^1$. Then up to one exception $z' \in \mathbb{P}^1$*

there is a lattice translate of $\operatorname{tail}(P)$ *in* $\mathcal{S}_y$, *for every* $y \neq z$.

Proof. We denote the tail cone of P by τ. Consider the part R of $\mathcal{S}_z$ consisting of all maximal polyhedra with tail τ. We are looking at the boundary facets of this part. There is a facet having tail τ, it corresponds to a primitive lattice element $u \in \tau^\perp$, which is minimized on this facet. On the other side of the facet we have a neighboring full-dimensional polyhedron P' having a tail cone $\tau' \succ \tau$. Replacing P by P' and iterating this procedure, we end up with a maximal polyhedron P, a non-maximal tail cone $\tau = \operatorname{tail} P$, a region R of $\mathcal{S}_z$, and a facet of R minimizing some $u \in \tau^\perp$ (which necessarily has tail cone τ) such that the neighboring polyhedron Δ^+ has full-dimensional tail σ^+. Now, we treat two cases separately: **1)** $\dim \tau < n - 1$ and **2)** $\dim \tau = n - 1$.

In the first case, the common facet of Δ^+ and R has dimension $n-1$, but tail cone τ of dimension less than $n-1$. This implies that the facet and, hence, Δ^+ has at least $n - \dim\tau > 1$ vertices. In particular, it is not a lattice translate of a cone and by Lemma 6 the tail cone σ^+ has to be marked. Again by Lemma 6 for $y \neq z$ all but one of the $\mathcal{S}_y(\sigma^+)$ are lattice translate of σ^+. Hence, the faces of these $\mathcal{S}_y(\sigma^+)$ with tail cone τ are indeed lattice translates of τ and the claim is proved.

In the second case, $-u$ is minimized on another facet of R. For the neighboring full-dimensional polyhedron Δ^- we have $\tau \prec \sigma^- := \operatorname{tail}\Delta^-$. Since τ is of dimension $n-1$, the cone σ^- must be full-dimensional. By construction $\sigma^+ \cap \sigma^- = \tau$. Assume that σ^+ is not marked. Then all polyhedra $\mathcal{S}_y(\sigma^+) \in \mathcal{S}_y$ are lattice translations of σ^+. As before, we infer that the claim is fulfilled in this case. The same applies if σ^- is not marked.

Now assume that both σ^+ and σ^- are marked. There are p-divisors in $\mathcal{D}^+, \mathcal{D}^- \in \mathcal{S}$ with $\operatorname{tail} \mathcal{D}^\pm = \sigma^\pm$ and $\deg \mathcal{D}^\pm \neq \emptyset$. If $\Delta^\pm = \mathcal{D}^\pm_z$ is not a lattice translate, then we know that all other polyhedra $\mathcal{D}^\pm_y$ are lattice translates of $\sigma^\pm$ up to one exception. Hence, every $\mathcal{D}^\pm_y$ up to one exception contains a lattice translation of $\tau \prec \sigma^\pm$ and the claim follows. Hence, we may assume that $\mathcal{D}^+_z$, $\mathcal{D}^-_z$ are just lattice translates of the cone σ^+ and σ^- respectively.

Remember that we have a maximal polyhedron $P \in \mathcal{S}_z$ with non-maximal tail cone τ. Hence, there is some p-divisor $\mathcal{D}(P) \in \mathcal{S}$ with $\mathcal{D}(P)_z = P$. By the properness condition (3) we have $\deg \mathcal{D}(P) = \emptyset$ and by the degree rule we have $\tau \cap \deg \mathcal{D}^\pm = \emptyset$. Now, by Corollary 2 we know that all $\mathcal{D}^\pm_y$ are just translated cones $(v^\pm_y + \sigma^\pm)$. Moreover, up to two exceptions $\mathcal{D}^\pm_{y_0} = (v^\pm_0 + \sigma)$ and $\mathcal{D}^\pm_{y_\infty} = (v^\pm_\infty + \sigma)$ they are even lattice translates, i.e. $v^\pm_y \in N$.

Corollary 3 ensures that τ is a regular cone. Hence, the primitive ray generators $e_1, \dots e_{n-1}$ of τ form a part of a basis $e_1, \dots e_n$ of N. Since $u \in \tau^\perp$ we have $\langle u, e_n \rangle = 1$. Now, the elements $(e_i, 0)$ together with $(0, 1)$ form a basis of $N \times \mathbb{Z}$. We use this basis for an identification $N \times \mathbb{Z} \cong \mathbb{Z}^{n+1}$. In particular, $\langle u, \cdot \rangle$ equals to the n-th coordinate in this basis.

By Lemma 5, the primitive ray generators of $\operatorname{cone}(w_0^\pm + \mathcal{D}_{y_0}^\pm, w_\infty^\pm + \mathcal{D}_{y_\infty}^\pm)$ (as in Example 3) are given by the columns of the following matrix. Due to the smoothness condition these matrices have to be unimodular. There first $n-1$ columns correspond to the rays of τ and the last two columns to the vertex in $\mathcal{D}_{y_0}$ and $\mathcal{D}_{y_\infty}$, respectively.

$$M^\pm = \begin{pmatrix} 1 & & & * & * \\ & \ddots & & \vdots & \vdots \\ & & 1 & * & * \\ 0 & \cdots & 0 & \langle v_\infty^\pm + w_\infty^\pm, u \rangle & \langle v_\infty^\pm + w_\infty^\pm, u \rangle \\ 0 & \cdots & 0 & \mu_0^\pm & -\mu_\infty^\pm \end{pmatrix}$$

Here, $\mu_0^\pm$, $\mu_\infty^\pm$ are minimal positive integers such that $\mu_0^\pm \cdot v_0^\pm$ and $\mu_\infty^\pm \cdot v_\infty^\pm$ are lattice elements. By the slice rule, we have $\langle u, v_y^+\rangle \geq \langle u, v_y^-\rangle$ (else $(v_y^+ + \sigma^+)$ and $(v_y^- + \sigma^-)$ would intersect in a non-face, since $\tau = \sigma^+ \cap u^\perp = \sigma^- \cap u^\perp$ is a common facet). Moreover, $\langle v_z^+, u\rangle > \langle v_z^-, u\rangle$ holds, since $\Delta^+ = (v_z^+ + \sigma^+)$ and $\Delta^- = (v_z^- + \sigma^-)$ are separated by the full-dimensional region R. Note that the compared values are integers. Let us set $\Sigma^\pm = \sum_y v_y^\pm$. By definition, we have $v^\pm = \Sigma^\pm - v_0^\pm - v_\infty^\pm$. We obtain $\langle \Sigma^+, u\rangle \geq \langle \Sigma^-, u\rangle + 1$.

We choose w_0^+ in a way such that $0 \leq \langle v_0^+ + w_0^+, u\rangle < 1$ holds and set $w_\infty^+ = v^+ - w_0^+$, $w_\infty^- = w_\infty^+ - \lfloor v_\infty^- - v_\infty^+ \rfloor$ (componentwise rounding) and $w_0^- = v^- - w_\infty^-$. Hence, we obtain $\langle v_\infty^- + w_\infty^-, u\rangle \leq \langle v_\infty^+ + w_\infty^+, u\rangle$ and

$$\begin{aligned}
v_0^- + w_0^- &= v_0^- + v^- - w_\infty^- = \Sigma^- - v_\infty^- - w_\infty^- \\
&= \Sigma^- - v_\infty^- - w_\infty^+ + \lfloor v_\infty^- - v_\infty^+ \rfloor \\
&= \Sigma^- - v_\infty^- - v^+ + w_0^+ + \lfloor v_\infty^- - v_\infty^+ \rfloor \\
&= \Sigma^- - v_\infty^- - \Sigma^+ + v_0^+ + v_\infty^+ + w_0^+ + \lfloor v_\infty^- - v_\infty^+ \rfloor \\
&= w_0^+ + v_0^+ + (\Sigma^- - \Sigma^+) + \left(\lfloor v_\infty^- - v_\infty^+ \rfloor - (v_\infty^- - v_\infty^+)\right).
\end{aligned}$$

After pairing with u we obtain $\langle v_0^- + w_0^-, u\rangle \leq \langle w_0^+ + v_0^+, u\rangle - 1 < 0$. Hence, either $\langle v_0^+ + w_0^+, u\rangle,\ \langle v_\infty^+ + w_\infty^+, u\rangle \geq 0$ or $\langle v_0^- + w_0^-, u\rangle,\ \langle v_\infty^- + w_\infty^-, u\rangle \leq 0$. In both cases we need to have either $\mu_0^\pm = 1$ or $\mu_\infty^\pm = 1$ in order to obtain $|\det M^\pm| = 1$. All but one coefficient of $\mathcal{D}^+$ or $\mathcal{D}^-$, respectively, are lattice translates. Since τ is a face of $\sigma^\pm$ we will always find a lattice translate of τ as well, and Proposition 7 is proved. □

Proof of Theorem 5. Consider a set $\mathcal{S}$ of p-divisors giving rise to a covering of X as above. We construct another set of p-divisors $\mathcal{S}'$ giving rise to an A-covering of X.

Let σ be a marked maximal cone in $\operatorname{tail}\mathcal{S}$. There is a $\mathcal{D} \in \mathcal{S}$ with $\deg\mathcal{D} \neq \emptyset$ and $\operatorname{tail}\mathcal{D} =$

σ. We simply add it to $\mathcal{S}'$. By Lemma 6, $\mathfrak{X}(\mathcal{D})$ is an affine space. If σ is maximal but not marked, then by Lemma 6 the polyhedra $\mathcal{S}_y(\sigma)$ are just lattice translates of σ. Now, we add the following two polyhedral divisors to $\mathcal{S}'$:

$$\mathcal{D}_0 = \emptyset \cdot 0 + \sum_{y \neq 0} \mathcal{S}_y(\sigma) \cdot y,$$

$$\mathcal{D}_\infty = \emptyset \cdot \infty + \sum_{y \neq \infty} \mathcal{S}_y(\sigma) \cdot y.$$

From Example 3 we know that $\mathfrak{X}(\mathcal{D}_0)$ and $\mathfrak{X}(\mathcal{D}_\infty)$ are both affine spaces.

By these considerations $\mathcal{S}'_y$ covers all polyhedra from $\mathcal{S}_y$ having maximal tail cones. Moreover, the markings are the same as for $\mathcal{S}$. It remains to care for maximal polyhedra P having non-maximal tail τ. We consider such a polyhedron living in some slice $\mathcal{S}_z$. By Proposition 7, we have a lattice translate $(v_y + \tau)$ in every slice except for $\mathcal{S}_z$ and

$\mathcal{S}_{z'}$. Having this, we can add the p-divisor $\mathcal{D}(P) = \emptyset \cdot z' + P \cdot z + \sum_{y \neq z,z'} (v_y + \tau) \cdot y$ to $\mathcal{S}'$. Thus for all maximal polyhedra with non-maximal tail we obtain $\mathcal{S}_y = \mathcal{S}'_y$ for all $y \in \mathbb{P}^1$. From Example 3 we know that $\mathfrak{X}(\mathcal{D}(P))$ are affine spaces. Hence, we obtain an A-covering of X. $\square$

Example 5. Let us illustrate the proof for the slices in Example 4. Here all the maximal tail cones are marked. Hence, $\mathcal{S}'$ contains a p-divisor of non-empty degree for every maximal tail cone. The polyhedral coefficient can be read off directly from the pictures. There remains a single maximal polyhedron $\sqcup$ with tail cone $\mathbb{Q}_{\geq 0} \cdot (0, 1)$. For $u = (1, 0)$ we find facets minimizing u and $-u$, respectively. The neighboring maximal polyhedra are just lattice translates of maximal tail cones, which are marked in our case. As stated in the proof, there is only one slice, where we do not find a lattice translates of one of the tail cones. This is the ∞-slice on the picture. Hence, we may add the p-divisor $\sqcup \otimes 0 + \emptyset \otimes \infty$ to $\mathcal{S}'$. By Example 3, this p-divisor corresponds to an affine space. Moreover, the slices of $\mathcal{S}'$ equal the given ones. Hence, we constructed a covering of the blow up of the quadric by six affine spaces.

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