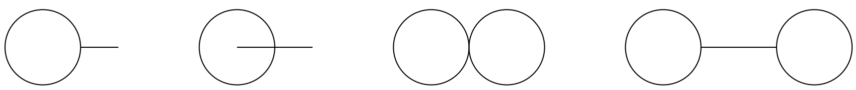
Coursework Solution / Feedback

1. Prove that no two of the following four path-connected subsets of the plane are homeomorphic. For the subsets (i) and (ii) you should assume that the end points of the line intervals are included in the subset and for the subset (iii) you should assume that the two circles are touching at a single point.



(i) (ii) (iii) (iv)

[8 marks]

Solution The existence of a cut point of order 3 distinguishes (ii) from all other subsets. Moreover, (iii) has only one cut point of order 2 while all the other subsets have infinitely many. [4 marks]

To distinguish (i) from (iv) one needs more subtle arguments. Consider the cut pairs of order 1, i.e. those, which leave a path-connected space after removing them. In (i) all these pairs share the end of the line segment as a common point. In (iv) one can choose an arbitrary point in each of the two circles (except from the intersection points with the line segment) to get infinitely many disjoint cut pairs.
Now assume that f is a homeomorphism between (i) and (iv) then it will take 1-pairs to one pair as shown in Question 1.12 of the exercises. However, since f is bijective it will take disjoint pairs to disjoint pairs. Hence, such a homeomorphism cannot exist. [4 marks]

2. (a) Determine which of the following collections of subsets give a topology on the set $X = \{a, b, c, d\}$, justifying your answers:

(i) $\tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, c, d\}, \{b, c, d\}, \{c, d\}\}$;

(ii) $\tau_2 = \{\emptyset, X, \{a, b\}, \{b, c, d\}\}$;

(iii) $\tau_3 = \{\emptyset, X, \{a\}, \{a, b, d\}, \{a, c\}, \{c\}\}$;

(b) If a collection of subsets in (a) does not give a topology, state without proof which subset or subsets must be added to the collection in order to give a topology.

(c) Consider the topological space $X = \mathbb{Z}$ with topology τ given by the basis
$$\{\{n\} \mid n \text{ is odd}\} \cup \{\{n-1, n, n+1\} \mid n \text{ is even}\}.$$
Show, that this topological space is path-connected. If you have trouble doing so, start with proving that there is a path between 0 and 1.

[11 marks]

Solution

(a) τ_1 is not a topology since $\{a, b\} = \{a\} \cup \{b\}$ is not included. τ_2 is not a topology since $\{b\} = \{a, b\} \cap \{b, c, d\}$ is not included. τ_3 is a topology. [5 marks]

(b) Adding $\{a, b\}$ to τ_1 and $\{b\}$ to τ_2 gives topologies on X.

[2 marks]

(c) We define the map $f\colon \mathbb{R} \to X$ by
$$f(t) = \begin{cases} t & \text{if } t \text{ is an even integer,} \\ \text{the odd integer which is closest to } t & \text{otherwise.} \end{cases}$$
This is a continuous map. It is sufficient to test this on the basis. Now, if n is odd we have $f^{-1}(\{n\}) = (n-1, n+1)$, which is open ind $\mathbb{R}$. For even n we have
$$f^{-1}(\{n-1, n, n+1\}) = (n-2, n-1) \cup \{n\} \cup (n+1, n+2) = (n-2, n+2),$$
which is again open in $\mathbb{R}$. The map $f : \mathbb{R} \to X$ is a continuous surjection from a path-connected topological space. Hence, X is path-connected as well. Alternatively this can be proved by constructing an explicit path between n and $n+1$ and then using induction.

[4 marks]

3. The aim of this problem is to prove that S^3 is homeomorphic to two copies of the solid torus glued along their respective boundaries.
Let
$$T = S^1 \times D^2 = \left\{(z, w) \in \mathbb{C}^2 \mid |z|^2 = 1,\ |w|^2 \leqslant 1\right\},$$
be the solid torus. The usual (hollow) torus $S^1 \times S^1$ can be seen as its boundary.

Consider two disjoint copies of the solid torus

$$T_0 = T \times \{0\} \subset \mathbb{C}^2 \times \mathbb{R}, \qquad T_1 = T \times \{1\} \subset \mathbb{C}^2 \times \mathbb{R}.$$

Let $\sim$ denote the equivalence relation on $T_0 \cup T_1$ generated by

$$(t_1, t_2, 0) \sim (t_2, t_1, 1) \quad \text{for} \quad (t_1, t_2) \in S^1 \times S^1.$$

Your task is show, that

$$(T_0 \cup T_1)/{\sim} \quad \cong \quad S^3 = \{(z, w) \in \mathbb{C}^2 \mid \ |z|^2 + |w|^2 = 1\}.$$

For this find bijective continuous functions

$$\begin{aligned} f_0 \colon T_0 &\to \{(z, w) \in S^3 \mid \ |z| \geqslant |w|\} \subset S^3, \\ f_1 \colon T_1 &\to \{(z, w) \in S^3 \mid \ |z| \leqslant |w|\} \subset S^3, \end{aligned}$$

which induces a homeomorphism $F \colon (T_0 \cup T_1)/\sim \to S^3$ as follows [without using the statements of either Theorem 0.20 or Theorem 3.18].

(a) Show that

$$\begin{aligned} f_0(z, w, 0) &= \frac{1}{\sqrt{|z|^2 + |w|^2}}(z, w) \\ f_1(z, w, 1) &= \frac{1}{\sqrt{|z|^2 + |w|^2}}(w, z) \end{aligned}$$

define bijective continuous maps $f_0 \colon T_0 \to \{(z, w) \in S^3 \mid \ |z| \geqslant |w|\}$ and $f_1 \colon T_1 \to \{(z, w) \in S^3 \mid \ |z| \leqslant |w|\}$.

(b) Explain how this induces a continuous function $f \colon T_0 \cup T_1 \to S^3$.

(c) Explain how this induces a function $F \colon (T_0 \cup T_1)/{\sim} \to \ S^3$, and why it is well defined.

(d) Prove that F is a bijection.

(e) Prove that F is continuous using the universal property of the quotient topology.

(f) Prove that F^{-1} is continuous.
[Hint: You may find the Gluing Lemma useful here.]

[11 marks]

Solution

(a)

$$f_0(z,w,0) = \frac{1}{\sqrt{|z|^2+|w|^2}}(z,w)$$
$$f_1(z,w,1) = \frac{1}{\sqrt{|z|^2+|w|^2}}(w,z)$$

One has

$$\left|\frac{z}{\sqrt{|z|^2+|w|^2}}\right|^2 + \left|\frac{w}{\sqrt{|z|^2+|w|^2}}\right|^2 = \frac{|z|^2+|w|^2}{|z|^2+|w|^2} = 1.$$

Hence, the images lies indeed in S^3. Moreover, rescaling both components of the vector (z,w) by a positive constant does to change the fact that $|z| \geqslant |w|$ from the beginning.

The functions f_0 and f_1 are bijective with inverses given by

$$f_0^{-1}(u,v) = \left(\frac{u}{|u|}, \frac{v}{|u|}, 0\right)$$
$$f_1^{-1}(u,v) = \left(\frac{v}{|v|}, \frac{u}{|v|}, 1\right)$$

For this one checks, that $f_j \circ f_j^{-1}$ and $f_j^{-1} \circ f_j$ for $j = 0,1$ both give the identity on their respective domains. The inverses are obviously continuous. [3 marks]

(b) By Gluing Lemma this defines a function

$$f(z,w,r) = \begin{cases} f_0(z,w,0) & r=0 \\ f_1(z,w,1) & r=1 \end{cases}$$

Indeed T_0 and T_1 are closed. The easiest way to see this is to indentify T_j as the preimage of $\{j\} \in \mathbb{R}$ under the projection to the second factor. Since $\{j\} \subset \mathbb{R}$ is closed we see that T_0 and T_1 are closed. The second assumption of the GLuing Lemma is trivially fulfilled as $T_0 \cap T_1 = \emptyset$. [2 mark]

(c) Note, that $f(z,w,r) = f(z',w',r')$ if and only if $(z,w,r) \sim (z',w',r')$ as defined above. We may define $F\colon (T_0 \cup T_1)/{\sim} \to S^3$ via $F([x]) = f(x)$. With this F is well-defined as for $x' \sim x$ we have $f(x') = f(x)$. Hence, the definition of $F([x])$ does not depend on the choice of a representative. [2 mark]

(d) We an inverse for F is given by the following formula

$$F^{-1}(u,v) = \begin{cases} (q \circ f_0^{-1})(u,v) & |u| \geqslant |v| \\ (q \circ f_1^{-1})(u,v) & |u| \leqslant |v| \end{cases}$$

With this F^{-1} is indeed well-defined as for $|u| = |v|$ we have $f_0^{-1}(u,v), f_1^{-1}(u,v) \in S^1 \times S^1$ and moreover $f_0^{-1}(u,v) \sim f_1^{-1}(u,v)$. Hence, $[f_0^{-1}(u,v)] = [f_1^{-1}(u,v)]$. [2 mark]

(e) We have $F \circ q(x) = F([x]) = f(x)$ by the definition of F. Now the universal property says that F is continuous if and only if $f = F \circ q$ is so. But f is continuous as shown in (b). [1 mark]

(f) The formula for F^{-1} is piecewise defined on two closed subsets of S^3. The formulae for the two pieces give continuous functions as q and f_i^{-1} are both continuous.

We have seen that on the intersection both formulae coincide. Morover,

$$g^{-1}([0,1]) = \{(u,w) \in S^3 \mid |u| \geqslant |v|\},$$
$$g^{-1}([-1,0]) = \{(u,w) \in S^3 \mid |u| \leqslant |v|\},$$

where $g(u,v) = |u| - |v|$. This function is continuous and, hence, both subsets are closed. Hence, by Gluing Lemma the inverse is a continuous function.

[1 mark]

For an excellent visual description of this gluing process have a look at the following answer to a question on stackexchange:
math.stackexchange.com/a/1075163.