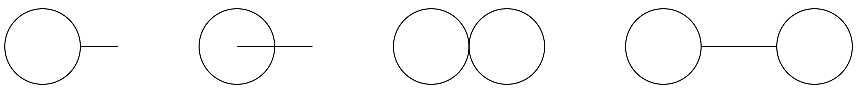
Coursework 2021

to be handed in online before 13:00 on 16 April 2021

1. Prove that no two of the following four path-connected subsets of the plane are homeomorphic. For the subsets (i) and (ii) you should assume that the end points of the line intervals are included in the subset and for the subset (iii) you should assume that the two circles are touching at a single point.



(i) (ii) (iii) (iv)

[8 marks]

2. (a) Determine which of the following collections of subsets give a topology on the set $X = \{a, b, c, d\}$, justifying your answers:

(i) $\tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, c, d\}, \{b, c, d\}, \{c, d\}\}$;

(ii) $\tau_2 = \{\emptyset, X, \{a, b\}, \{b, c, d\}\}$;

(iii) $\tau_3 = \{\emptyset, X, \{a\}, \{a, b, d\}, \{a, c\}, \{c\}\}$;

(b) If a collection of subsets in (a) does not give a topology, state without proof which subset or subsets must be added to the collection in order to give a topology.

(c) Consider the topological space $X = \mathbb{Z}$ with topology τ given by the basis

$$\{\{n\} \mid n \text{ is odd}\} \cup \{\{n-1, n, n+1\} \mid n \text{ is even}\}.$$

Show, that this topological space is path-connected. If you have trouble doing so, start with proving that there is a path between 0 and 1.

[11 marks]

3. The aim of this problem is to prove that S^3 is homeomorphic to two copies of the solid torus glued along their respective boundaries.

Let

$$T = S^1 \times D^2 = \left\{(z, w) \in \mathbb{C}^2 \mid |z|^2 = 1,\ |w|^2 \leqslant 1\right\},$$

be the solid torus. The usual (hollow) torus $S^1 \times S^1$ can be seen as its boundary.
Consider two disjoint copies of the solid torus

$$T_0 = T \times \{0\} \subset \mathbb{C}^2 \times \mathbb{R}, \qquad T_1 = T \times \{1\} \subset \mathbb{C}^2 \times \mathbb{R}.$$

Let $\sim$ denote the equivalence relation on $T_0 \cup T_1$ generated by

$$(t_1, t_2, 0) \sim (t_2, t_1, 1) \quad \text{for} \quad (t_1, t_2) \in S^1 \times S^1.$$

Your task is show, that

$$(T_0 \cup T_1)/{\sim} \quad \cong \quad S^3 = \{(z, w) \in \mathbb{C}^2 \mid \ |z|^2 + |w|^2 = 1\}.$$

For this find bijective continuous functions

$$f_0 \colon T_0 \to \{(z, w) \in S^3 \mid \ |z| \geqslant |w|\} \subset S^3,$$
$$f_1 \colon T_1 \to \{(z, w) \in S^3 \mid \ |z| \leqslant |w|\} \subset S^3,$$

which induces a homeomorphism $F \colon (T_0 \cup T_1)/\sim \to S^3$ as follows [without using the statements of Theorem 0.20, Theorem 3.18 or Theorem 5.10].

(a) Show that

$$f_0(z, w, 0) = \frac{1}{\sqrt{|z|^2 + |w|^2}}(z, w)$$
$$f_1(z, w, 1) = \frac{1}{\sqrt{|z|^2 + |w|^2}}(w, z)$$

define bijective continuous maps $f_0 \colon T_0 \to \{(z, w) \in S^3 \mid \ |z| \geqslant |w|\}$ and $f_1 \colon T_1 \to \{(z, w) \in S^3 \mid \ |z| \leqslant |w|\}$.

(b) Explain how this induces a continuous function $f \colon T_0 \cup T_1 \to S^3$.

(c) Explain how this induces a function $F \colon (T_0 \cup T_1)/{\sim} \to \ S^3$, and why it is well defined.

(d) Prove that F is a bijection.

(e) Prove that F is continuous using the universal property of the quotient topology.

(f) Prove that F^{-1} is continuous.
[Hint: You may find the Gluing Lemma useful here.]

[11 marks]