1.

(a) Define what is meant by a *topology* on a set X.

(b) Define what is meant by saying that a function $f\colon X \to Y$ between topological spaces is *continuous*. Define what is meant by saying that f is a *homeomorphism*.

(c) Consider the set $X = \{1, 2, 3\}$ and $\tau = \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}\}$. Show that τ defines a topology on X. For each bijection $X \to X$ decide whether it is a homeomorphism.

[13 marks]

Solution

(a) Given a set X, a *topology* on X is a collection τ of subsets of X with the following properties:

(i) $\emptyset \in \tau$, $X \in \tau$;

(ii) the intersection of any two subsets in τ is in τ:
$$U_1, U_2 \in \tau \Rightarrow U_1 \cap U_2 \in \tau;$$

(iii) the union of any collection of subsets in τ is in τ:
$$U_\lambda \in \tau \text{ for all } \lambda \in \Lambda \Rightarrow \bigcup_{\lambda \in \Lambda} U_\lambda \in \tau.$$

[6 marks, bookwork]

(b) $f\colon X \to Y$ is *continuous* if
$$V \text{ is open in } Y \Rightarrow f^{-1}(V) \text{ is open in } X$$

[1 marks, bookwork]

A *homeomorphism* is a continuous bijection with continuous inverse.

[2 marks, bookwork]

(c) $\emptyset$ and X are clearly contained in τ. Since $\emptyset \subset \{1\} \subset \{1, 2\}, \subset \{1, 2, 3\}$ every intersection or union will coinside with the largest or smallest, respectively, of the involed subsets. If $f\colon X \to X$ is a bijection, then it will map a subset to a subset of the same cardinality. On the other hand, a homeomorphism maps open subsets to open subsets. It follows that $f(\{1\}) = \{1\}$. Hence, $f(1) = 1$. Moreover, $f(\{1, 2\}) = \{1, 2\}$. Since, we already know, that $f(1) = 1$ we because of bijectivity must have $f(2) = 2$. Again by bijectivity of f it follows that $f(3) = 3$ and $\sigma = \mathrm{id}_X$. [4 marks, bookwork]

[Total: 13 marks]

The question was generally done well with the exception of finding the homeomorphisms. Here, a lot of things went wrong. Some students considered all permutations to be homeomorphisms. Others (contrary to the given definitions in (b)) considered homeomorphisms as maps on the topology. They are maps on the underlying sets. Hence, to state a homeomorphism one needs to write down a map $f\colon X \to X$.

2.

(a) Define what is meant by saying that a topological space X is *path-connected.*

(b) What is meant by saying that path-connectedness is a *topological property*?

(c) Define what is meant by the set $\pi_0(X)$ of *path-components* of a topological space X.

(d) Prove that a continuous map of topological spaces $f\colon X \to Y$ induces a map $f_*\colon \pi_0(X) \to \pi_0(Y)$ between the sets of path-components, taking care to prove that your function is well-defined. Prove that if f is a homeomorphism then f_* is a bijection.

(e) Use (d) to show that path-connectedness is a topological property.

(f) Show that $\mathbb{R} \setminus \{0\}$ has exactly two path-components, i.e. $\#\pi_0(\mathbb{R} \setminus \{0\}) = 2$,

[15 marks]

Solution

(a) A *path* from x_0 to x_1 in X is a continuous function $\sigma\colon [0,1] \to X$ with $\sigma(0) = x_0$ and $\sigma(1) = x_1$. X is said to be *path-connected* if, for each pair of points $x_0, x_1 \in X$, there is a path in X from x_0 to x_1.

[3 marks, bookwork]

(b) Saying that path-connectedness is a *topological property* means that, if $X \cong Y$ are homeomorphic topological spaces, then X is path connected if and only if Y is path-connected.

[1 marks, bookwork]

(c) Define an equivalence relation on X by $x \sim x'$ if and only if there is a path in X from x to x'. Then the path-components of X are the equivalence classes and $\pi_0(X)$ is the set of all equivalence classes.

[2 marks, bookwork]

(d) Suppose that $f\colon X \to Y$ is a continuous map. Then this induces a function $f\colon \pi_0(X) \to \pi_0(Y)$ by $f([x]) = [f(x)]$. This is well-defined because $[x] = [x']$ implies that $x \sim x'$ so that there is a path $\sigma\colon [0,1] \to X$ in X from x to x'. Then $f \circ \sigma\colon [0,1] \to Y$ is a path in Y from $f(x)$ to $f(x')$ and so $[f(x)] = [f(x')]$.

[3 marks, bookwork]

If f is a homeomorphism then f_* is a bijection since the inverse $g = f^{-1}\colon Y \to X$ induces a function $g_* : \pi_0(Y) \to \pi_0(X)$ inverse to f_* since $g_*(f_*([x])) = [g(f(x))] = [x]$ and $f_*(g_*([y])) = [y]$.
[2 marks, bookwork]

(e) Being path-connected is by definition equivalent to $\#\pi_0(X) = 1$. Hence, this follows directly from (d).
[1 marks, bookwork]

(f) The path-components are $(-\infty, 0)$ and $(0, \infty)$. To show this consider two points $x_0, x_1 < 0$ then
$$\sigma\colon [0,1] \to \mathbb{R} \setminus \{0\}; t \mapsto tx_1 + (1-t)x_0$$
is a path from x_0 to x_1. Note, that $\sigma(t) < 0$ and, hence, in $\mathbb{R} \setminus \{0\}$. It follows that $x_0 \sim x_1$ and $[x_0] = [x_1]$. Similarly, for $x_0, x_1 > 0$. On the other hand, for $x_0 < 0$ and $x_1 > 0$ there is no path in $\mathbb{R} \setminus \{0\}$ by intermediate value theorem. [3 marks, bookwork]

[Total: 15 marks]

In (d) some people forgot to check that their maps are indeed well defined. Another problem was that students confused this with the similar map defined on the fundamental group. For (f) many people only stated the path-components of $\mathbb{R} \setminus \{0\}$. But you also needed to show that these are ideed the path-components i.e. I wanted to see you applying the definition from (a)

3.

(a) Define what is meant by saying that a topological space is *Hausdorff*.

(b) Which of the following spaces are Hausdorff? Justify your answers.

- (i) $\mathbb{Z}$ with the discrete topology $\tau_1 = \mathcal{P}(X)$,
- (ii) $\mathbb{Z}$ with the indiscrete topology $\tau_2 = \{\emptyset, \mathbb{Z}\}$,
- (iii) $\mathbb{Z}$ with the co-finite topology $\tau_3 = \{U \subset \mathbb{Z} \mid \mathbb{Z} \setminus U \text{ is finite}\} \cup \{\emptyset, \mathbb{Z}\}$.

(c) Suppose that X and Y are topological spaces. Define the *product topology* on the Cartesian product $X \times Y$. [It is not necessary to prove that this is a topology.]

(d) Let $\Delta = \{(x, x) \mid x \in X\} \subset X \times X$ be the diagonal. Prove that if Δ is closed in the product topology, then X is Hausdorff.

[15 marks]

Solution

(a) The topological space X is *Hausdorff* if, for each distinct pair of points x, $y \in X$, there exist open sets U and V in X such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

[2 marks, bookwork]

(b)
- (i) this space is Haussdorff, since for every two elements $a, b \in \mathbb{Z}$ the singletons $\{a\}$ and $\{b\}$ provide the disjoint open neighbourhoods.
- (ii) this space is not Hausdorff, as the only choice for an open neighbourhood is $\mathbb{Z}$ itself. Hence, any two open neighbourhoods are actually equal, hence, in particular not disjoint.
- (iii) this space is not Haussdorff, as any two open neighbourhoods can differ in at most finitely may points.

[6 marks, bookwork]

(c) The product topology on $X \times Y$ has a basis

$$\{U \times V \mid U \text{ open in } X, V \text{ open in } Y\},$$

i.e. the open sets consist of all unions of such sets.

[3 marks, bookwork]

(d) Assume Δ is closed. Hence $X \times X \setminus \Delta$ is open. By definition of the product topology this means it is a union of open rectangles, i.e. sets of the form $U \times V \subset X \times X \setminus \Delta$ with U and V both open in X. Consider $x, y \in X$ with $x \neq y$ then (x, y) lies outside the diagonal. Hence, is has to be contained in such a set

$$U \times V \subset X \times X \setminus \Delta.$$

On the one hand this implies that $x \in U$ and $y \in V$. On the other hand $U \cap V = \emptyset$, since for $x \in U \cap V$ one would have $\Delta \ni (x, x) \in U \times V$. [4 marks, question set]

[Total: 15 marks]

A common mistake in (c) was to state only the basis of the topology but to claim that this is the whole topology.

4.

(a) Define what is meant by a *compact subset* of a topological space.

(b) Prove by using only the definition of compactness, that $[0, 1) \subset \mathbb{R}$ is not compact.

(c) Prove that every compact subset of a Hausdorff topological space is closed.

(d) Give an example showing that the Hausdorff condition in (b) cannot be omitted.

[15 marks]

Solution

(a) $K \subset X$ is compact if each cover of K by open subsets of X has a finite subcover.

[3 marks, bookwork]

(b) Consider the open subsets $U_s = [0, s)$ for $s \in (0, 1)$. Then $[0, 1) = \bigcup_{s \in (0,1)} U_s$, since for $t \in [0, 1)$, this t is contained in $U_{t+(1-t)/2}$. Now, assume there is a finite subcover $\{[0, s_1), \ldots, [0, s_\ell)\}$. Then $s := \max\{s_1, \ldots, s_\ell\} < 1$. Hence, $\bigcup_{i=1}^{\ell} U_{s_i} = [0, s) \subsetneq [0, 1)$.

[5 marks, bookwork]

(c) Suppose that K is a compact subset of a Hausdorff space X. To prove that K is closed we prove that $X \setminus K$ is open, and we prove this by proving that it is a union of open subsets. Let $x \in X \setminus K$. Then, by the Hausdorff condition, for each $a \in K$ there are open subsets U_a, V_a of X such that $a \in U_a$, $x \in V_a$ and $U_a \cap V_a = \emptyset$.

Then $\{U_a \mid a \in A\}$ is an open cover for K. Hence, since K is compact, there is a finite subcover $\{U_{a_i} \mid 1 \leqslant i \leqslant n\}$ for K. So $K \subset \bigcup_{i=1}^n U_{a_i}$.

Put $V_x = \bigcap_{i=1}^n V_{a_i}$. Then V_x is a finite intersection of open sets and so is open and, since $x \in V_{a_i}$ for all i, $x \in V_x$. Furthermore, for $1 \leqslant i \leqslant n$, $V_x \cap U_{a_i} \subset V_{a_i} \cap U_{a_i} = \emptyset$ and so $V_x \cap U_{a_i} = \emptyset$. Hence $V_x \cap \bigcup_{i=1}^n U_{a_i} = \emptyset$ and so $X_x \cap K = \emptyset$ or, equivalently, $V_x \subset X \setminus K$.

Thus $X \setminus K = \bigcup_{x \in X \setminus K} V_x$ is a union of open sets and so is open. Hence K is closed.

[5 marks, bookwork]

(d) Consider $X = [-1,1]/\sim$, where $s \sim \pm s$ when $|s| < 1$. Then $U^+ = X \setminus \{[-1]\} \cong [0,1]$ (via $q|_{[0,1]}$) is compact. But, $q^{-1}(U^+) = (-1,1]$, which is not closed in $[-1,1]$. Hence, U^+ is not closed. [2 marks, bookwork]

[Total: 15 marks]

Part (b) caused a lot of trouble. Some people didn't realise that they have to state a (necessarily infinite) open cover without finite subcover. Furthermore a common mistake was to consider open subsets like $(1/n, 1)$ for the cover. However, no union of such subsets will contain 1. In order to come up with the correct solution one should recognise, that the compactness is violated at the upper boundary of the interval. Part (c) was attempted only be a few students. I have seen many correct alternative solutions for (d), e.g. indiscrete spaces (here all subsets are compact but most of them not closed)

5.

(a) Suppose that $q\colon X \to Y$ is a surjection from a topological space X to a set Y. Define the *quotient topology* on Y determined by q. State the *universal property* of the quotient topology.

(b) Suppose that $f\colon X \to Z$ is a continuous surjection from a compact topological space X to a Hausdorff topological space Z. Define an equivalence relation $\sim$ on X so that f induces a bijection $F\colon X/\!\sim\, \to Z$ from the identification space $X/\!\sim$ of this equivalence relation to Z. Prove that F is a homeomorphism. [State clearly any general results which you use.]

(c) Prove that

$$f\colon [-1,1] \times \{-1,1\} \to S^1;\ (t,a) \mapsto (t, a \cdot \sqrt{1-t^2}).$$

induces a homeomorphism from the quotient space $([-1,1] \times \{-1,1\})/\!\sim$ to the circle $S^1 \subset \mathbb{R}^2$, where $\sim$ is generated by $(1,1) \sim (1,-1)$ and $(-1,1) \sim (-1,-1)$, i.e.

$$(t,a) \sim (t',a') \Leftrightarrow t = t' \text{ and either } a = a' \text{ or } |t| = 1.$$

[15 marks]

Solution

(a) Given a topological space (X, τ) and a surjection $q\colon X \to Y$ the quotient topology on Y is given by

$$\{V \subset Y \mid q^{-1}(V) \in \tau\}.$$

The *universal property* of the quotient topology is: $f\colon Y \to Z$ to a topological space Z is continuous if and only if the composition $f \circ q\colon X \to Z$ is continuous.

[4 marks, bookwork]

(b) Given a continuous surjection $f\colon X \to Z$, define an equivalence relation on X by $x \sim x' \Leftrightarrow f(x) = f(x')$. Then we may define $F\colon X/{\sim} \to Z$ by $F([x]) = f(x)$. Since $[x] = [x'] \Leftrightarrow x \sim x' \Leftrightarrow f(x) = f(x')$ (by the definition of the equivalence relation), the function F is well-defined. Since $F([x]) = F([x']) \Leftrightarrow f(x) = f(x') \Leftrightarrow x \sim x'$ (by the definition of the equivalence relation) it follows that $[x] = [x']$ and F is injective. Since f is a surjection, $y = f(x)$ for some $x \in X$ and so $y = F([x])$. Hence F is a surjection. This shows that $F\colon X/{\sim} \to Z$ is a bijection. The map $F : X/{\sim} \to Z$ is continuous by the universal property since $F \circ q = f$ which is given as continuous, where $q\colon X \to X/{\sim}$ is the quotient map given by $q(x) = [x]$.

The space $X/{\sim} = q(X)$ is compact since it is the continuous image of a compact set. Hence F is a homeomorphism since it is a continuous bijection from a compact space to a Hausdorff space.

[7 marks, bookwork]

(c) First note, that f is continuous as the component functions are continuous. It is surjective, since for $(x, y) \in S^1$ and $y \geqslant 0$ one has $(x, y) = f(x, 1)$ and similarly $(x, y) = f(x, -1)$ for $y \leqslant 0$. Now, S^1 is Hausdorff (a subset of Euclidean space) and $[-1, 1] \times \{-1, 1\}$ is compact (a closed and bounded subset of Euclidean space). Now the result follows from (b).

[4 marks, new]

[Total: 15 marks]

Parts (a) and (b) where generally done well. In (c) many people forgot to say that the equivalence relation from (b) gives exactly the equivalence relation stated in the question. Also some students forgot to mention that the spaces involved are compact/Hausdorff (and why).

6.

(a) Prove that, if the product $\sigma_0 * \tau_0$ of two paths σ_0 and τ_0 in a topological space X is defined and the paths σ_1 and τ_1 are homotopic to σ_0 and τ_0 respectively, then the product $\sigma_1 * \tau_1$ is defined and is homotopic to $\sigma_0 * \tau_0$.

(b) Explain how a continuous function $f\colon X \to Y$ induces a homomorphism $f_*\colon \pi_1(X, x_0) \to \pi_1(X, f(x_0))$. You should indicate why f_* is well-defined and why it is a homomorphism.

(c) Prove that, for topological spaces X and Y with points $x_0 \in X$, $y_0 \in Y$, there is an isomorphism of groups

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

[15 marks]

Solution

(a) Given homotopic paths $H\colon \sigma_0 \sim \sigma_1$ and $K\colon \tau_0 \sim \tau_1$ such that $\sigma_0 * \tau_0$ is defined. Then ${}_1(1) = \sigma_0(1) = \tau_0(0) = \tau_1(0)$ and so the product $\sigma_1 * \tau_1$ is defined.

Suppose that $H\colon \sigma_0 \sim \sigma_1$ and $K\colon \tau_0 \sim \tau_1$. Then we may define a homotopy $L\colon \sigma_0 * \tau_0 \sim \sigma_1 * \tau_1$ by

$$L(s,t) = \begin{cases} H(2s,t) & \text{for } 0 \leqslant s \leqslant 1/2 \text{ and } t \in I, \\ K(2s-1,t) & \text{for } 1/2 \leqslant s \leqslant 1 \text{ and } t \in I. \end{cases}$$

This is well defined since, for $s = 1/2$, $H(1,t) = x_1 = K(0,t)$. In addition, L is continuous by the Gluing Lemma since $[0,1/2] \times I$ and $[1/2,1] \times I$ are closed subsets of I^2

[5 marks, bookwork]

(b) The function f_* is defined by $f_*([\sigma]) = [f \circ \sigma]$. It is well-defined since, if $[\sigma_0] = [\sigma_1]$ then $\sigma_0 \sim \sigma_1$ and so there exists a homotopy $H\colon \sigma_0 \sim \sigma_1$. Then $f \circ H\colon I^2 \to Y$ gives a homotopy $f \circ \sigma_0 \sim f \circ \sigma_1$ and so $[f \circ \sigma_0] = [f \circ \sigma_1]$.

To see that f_* is a homomorphism suppose that $[\sigma]$, $[\tau] \in \pi_1(X, x_0)$. Then

$$f_*([\sigma][\tau]) = f_*([\sigma * \tau]) = [f \circ (\sigma * \tau)]$$

and

$$f_*([\sigma])f_*([\tau]) = [f \circ \sigma][f \circ \tau] = [(f \circ \sigma) * (f \circ \tau)]$$

and by writing out the formulae we see that $f \circ (\sigma * \tau) = (f \circ \sigma) * (f \circ \tau)\colon I \to Y$. Hence, $f_*([\sigma][\tau]) = f_*([\sigma])f_*([\tau])$.

[5 marks, bookwork]

(c) Let $p_1\colon X \times Y \to X$ and $p_2\colon X \times Y \to Y$ be the projection maps. The function

$$\pi_1\big(X \times Y, (x_0, y_0)\big) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

given by $\alpha \mapsto \big((p_1)_*(\alpha), (p_2)_*(\alpha)\big)$ is an isomorphism. To see this we write down the inverse. Given a loop σ_1 in X based at x_0 and a loop σ_2 in Y based at y_0 then we may define a loop σ in $X \times Y$ based at (x_0, y_0) by $\sigma(s) = \big(\sigma_1(s), \sigma_2(s)\big)$. Then $([\sigma_1], [\sigma_2]) \mapsto [\sigma]$ is well-defined and provides the necessary inverse. Hence the given function is an isomorphism of groups.

[5 marks, question set]

[Total: 15 marks]

Part (a) was generally done well. For part (b) people forgot tho check that f_* is actually well-defined. A lot of problems occurred in part (c) with writing down the isomorphism and/or its inverse.

END OF EXAMINATION PAPER