In-class problems: Week 2

1. Prove that any two closed intervals $[a_1, b_1] = \{ x \in \mathbb{R} \mid a_1 \leqslant x \leqslant b_1 \}$ and $[a_2, b_2]$ (where $a_i < b_i$) are homeomorphic by writing down the formula for a homeomorphism.

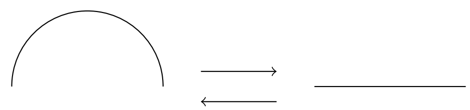
2. Let S^n denote the unit sphere in $\mathbb{R}^{n+1}$ with centre the origin

$$S^n = \{ \mathbf{x} \in \mathbb{R}^{n+1} \mid |\mathbf{x}| = 1 \}$$

and D^n denote the closed unit ball in $\mathbb{R}^n$ centre the origin

$$\mathbb{D}^n = \{ \mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x}| \leqslant 1 \}.$$

Prove that the closed semicircle $\{ \mathbf{x} \in S^1 \mid x_2 \geqslant 0 \}$ is homeomorphic to the closed interval $[-1, 1]$. More generally, prove that the closed hemisphere $\{ \mathbf{x} \in S^n \mid x_{n+1} \geqslant 0 \}$ is homeomorphic to the closed ball $\mathbb{D}^n$.



3. How does path-connectedness behave under the set operations $\cup$, $\cap$ and $\times$?
For $X, Y \subset \mathbb{R}^n$ does one of the statements below imply the other?

(a) X and Y are both path-connected.

(b) $X \square Y$ is path-connected.

Here, $\square$ stands for one of the operations $\cup$, $\cap$ and $\times$.

4. Use a cut-point argument to show that the three intervals $[0, 1]$, $(0, 1)$ and $[0, 1)$ are topologically distinct (i.e. no two are homeomorphic).