1 Topological Equivalence and Path-Connectedness

1.1 Definition. Suppose that X and Y are subsets of Euclidean spaces. A function $f\colon X \to Y$ is a *topological equivalence* or a *homeomorphism* if it is a continuous bijection such that the inverse $f^{-1}\colon Y \to X$ is also continuous. If such a homeomorphism exists then X and Y are *topologically equivalent* or *homeomorphic*, written $X \cong Y$.

1.2 Example. (a) The real line $\mathbb{R}$ and the open half line $(0, \infty) = \{\, x \in \mathbb{R} \mid x > 0 \,\}$ are homeomorphic. A homeomorphism is given by $\exp\colon \mathbb{R} \to (0, \infty)$ with inverse $\log_e\colon (0, \infty) \to \mathbb{R}$.

(b) $X = \mathbb{R}^2 \setminus \{\mathbf{0}\}$, the punctured plane, and $Y = \{\, \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = 1 \,\}$, the infinite cylinder, are homeomorphic. A homeomorphism $f\colon X \to Y$ is given by

$$f(x_1, x_2) = \big(x_1/|\mathbf{x}|, x_2/|\mathbf{x}|, \log_e(|\mathbf{x}|)\big)$$

with inverse $g\colon Y \to X$ given by

$$g(y_1, y_2, y_3) = e^{y_3}(y_1, y_2).$$

1.3 Exercise. (a) The punctured plane, $X = \mathbb{R}^2 \setminus \{\mathbf{0}\}$, is homeomorphic to the complement of the unit disc, $Z = \{\, \mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| > 1 \,\} = \mathbb{R}^2 \setminus D^2$ where $D^2 = \{\, \mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| \leqslant 1 \,\}$.

(b) $S^1 = \{\, \mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| = 1 \,\}$, the unit circle, and $T = \{\, \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \mid |x_1| + |x_2| = 1 \,\}$, the diagonal square, are homeomorphic.

1.4 Problem. We prove that two subsets are homeomorphic by writing down a homeomorphism. How can we prove that two subsets are *not* homeomorphic?

1.5 Definition. A property P of subsets of Euclidean spaces is a *topological property* when, if X and Y are homeomorphic subsets, then X has property P if and only if Y has property P.

Thus, if X has property P and Y does not have property P then X and Y are not homeomorphic.

Path-connected subsets of Euclidean space

1.6 Definition. (a) Let X be a subset of some Euclidean space. A *path* in X is a continuous function $\sigma\colon [0,1] \to X$ where $[0,1] = \{\, t \in \mathbb{R} \mid 0 \leqslant t \leqslant 1 \,\}$. The point $\sigma(0)$ is the *beginning point* of the path and the point $\sigma(1)$ is the *terminal point* of the path. We say that σ is a path in X *from* $\sigma(0)$ *to* $\sigma(1)$.

(b) The subset X is said to be *path-connected* if, for each pair of points $\mathbf{x}$, $\mathbf{x}' \in X$, there is a path in X from $\mathbf{x}$ to $\mathbf{x}'$.

1.7 Proposition. The closed unit ball (or disc) $D^n = \{\, \mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x}| \leqslant 1 \,\}$ in $\mathbb{R}^n$ is path-connected.

Proof. Given $\mathbf{x}$,$\mathbf{x}' \in D^n$ define $\sigma\colon [0,1] \to \mathbb{R}^n$ by

$$\sigma(s) = \mathbf{x} + s(\mathbf{x}' - \mathbf{x}) = (1-s)\mathbf{x} + s\mathbf{x}'$$

for $s \in [0,1]$. Then σ is continuous, $\sigma(0) = \mathbf{x}$ and $\sigma(1) = \mathbf{x}'$ so σ is a path in $\mathbb{R}^n$ from $\mathbf{x}$ to $\mathbf{x}'$.
However, for $0 \leqslant s \leqslant 1$, $|\sigma(s)| = |(1-s)\mathbf{x} + sx'| \leqslant |(1-s)\mathbf{x}| + |s\mathbf{x}'|$ (by the triangle inequality) $= (1-s)|\mathbf{x}| + s|\mathbf{x}'|$ (since $s \geqslant 0$ and $1 - s \geqslant 0$) $\leqslant (1-s) + s$ (since x, $\mathbf{x}' \in D^n$) $= 1$, i.e. $|\sigma(s)| \leqslant 1$. Hence $\sigma(s) \in D^n$ and so $\sigma\colon [0,1] \to D^n$ is a path in D^n from $\mathbf{x}$ to $\mathbf{x}'$.
Hence D^n is path-connected. □

1.8 Exercise. The unit circle S^1 in $\mathbb{R}^2$ is path-connected.

1.9 Theorem. Let $f\colon X \to Y$ be a continuous surjection where X and Y are subsets of Euclidean spaces. Then, if X is path-connected, so is Y.

Proof. Exercise. □

1.10 Corollary. Path-connectedness is a topological property.

Proof. Suppose that X and Y are homeomorphic subsets of Euclidean spaces. Then there is a homeomorphism $f\colon X \to Y$. Then if X is path-connected so is Y by the Theorem since f is a continuous surjection. Conversely, if Y is path-connected then so is X since $f^{-1}\colon Y \to X$ is a continuous surjection. Thus, X is path-connected if and only if Y is path-connected as required. □

1.11 Proposition. The subset $\mathbb{R} \setminus \{0\}$ is not path-connected and so $\mathbb{R} \setminus \{0\} \not\cong S^1$.

Proof. This is true because there is no path in $\mathbb{R} \setminus \{0\}$ from -1 to 1. This may be proved by contradiction. Suppose. for contradiction, that $\sigma\colon [0,1] \to \mathbb{R} \setminus \{0\}$ is a path from -1 to 1 so that $\sigma(0) = -1$ and $\sigma(1) = 1$. Then $i \circ \sigma\colon [0,1] \to \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is a continuous function with values -1 and 1 for which 0 is not a value. This contradicts the intermediate value property of the function σ (Theorem 0.23(b) in the Background Material) since $-1 < 0 < 1$ and so gives the necessary contradiction. Hence σ cannot exist, as required and so $\mathbb{R} \setminus \{0\}$ is not path-connected. It follows that $\mathbb{R} \setminus \{0\} \ncong S^1$ since S^1 is path-connected and path-connectedness is a topological property. $\square$

1.12 Problem. Are S^1 and $[0,1)$ homeomorphic? There is a continuous bijection $f\colon [0,1) \to S^1$ defined by $f(x) = (\cos 2\pi x, \sin 2\pi x)$.
More generally, is S^1 homeomorphic to any subset of $\mathbb{R}$?

Path-components

1.13 Definition. Suppose that X is a subset of a Euclidean space.

(a) Given $\mathbf{x} \in X$, we may define a path $\varepsilon_{\mathbf{x}} \colon [0,1] \to X$ by

$$\varepsilon_{\mathbf{x}}(s) = \mathbf{x} \quad \text{for } 0 \leqslant s \leqslant 1.$$

This is called the *constant path* at $\mathbf{x}$.

(b) Given a path $\sigma \colon [0,1] \to X$ in X we may define a path

$$\overline{\sigma}(s) = \sigma(1-s) \quad \text{for } 0 \leqslant s \leqslant 1.$$

This is called the *reverse path* of σ and is a path from $\sigma(1)$ to $\sigma(0)$.

(c) Given paths $\sigma_1 \colon [0,1] \to X$ and $\sigma_2 \colon [0,1] \to X$ in X such that $\sigma_1(1) = \sigma_2(0)$ we may define a path $\sigma_1 * \sigma_2 \colon [0,1] \to X$ by

$$\sigma_1 * \sigma_2(s) = \begin{cases} \sigma_1(2s) & \text{for } 0 \leqslant s \leqslant 1/2, \\ \sigma_2(2s-1) & \text{for } 1/2 \leqslant s \leqslant 1. \end{cases}$$

This is called the *product* of the paths σ_1 and σ_2 and is a path from $\sigma_1(0)$ to $\sigma_2(1)$.

[Note that $\sigma_1 * \sigma_2$ is well-defined and continuous at $t = 1/2$ by the conditions on σ_1 and σ_2.]

1.14 Proposition. Given X, a subset of a Euclidean space, we may define an equivalence relation on X as follows: for $\mathbf{x}$, $\mathbf{x}' \in X$, $\mathbf{x} \sim \mathbf{x}'$ if and only if there is a path in X from $\mathbf{x}$ to $\mathbf{x}'$.

Proof. We check the conditions for an equivalence relation (Definition 0.15).
The reflexive property. For each point $\mathbf{x} \in X$, $\mathbf{x} \sim \mathbf{x}$ using the constant path $\varepsilon_{\mathbf{x}}$.
The symmetric property. Suppose that $\mathbf{x}$ and $x' \in X$ such that $\mathbf{x} \sim \mathbf{x}'$. Then there is a path σ in X from $\mathbf{x}$ to $\mathbf{x}'$. The reverse path $\overline{\sigma}$ is then a path in X from $\mathbf{x}'$ to $\mathbf{x}$ and so $\mathbf{x}' \sim \mathbf{x}$ as requirec.
The transitive property. Suppose that $\mathbf{x}$, $\mathbf{x}'$ and $\mathbf{x}'' \in X$ such that $\mathbf{x} \sim \mathbf{x}'$ and $\mathbf{x}' \sim \mathbf{x}''$. This means that there is a paths σ_1 in X from $\mathbf{x}$ to $\mathbf{x}'$ and a path σ_2 in X from $\mathbf{x}'$ to $\mathbf{x}''$. Then the product path $\sigma_1 * \sigma_2$ is a path in X from $\mathbf{x}$ to $\mathbf{x}''$ and so $\mathbf{x} \sim \mathbf{x}''$ as required. $\square$

1.15 Definition. Given X, a subset of a Euclidean space, the equivalence classes of the equivalence relation in Proposition 1.14 are called the *path-components* of X. We write $\pi_0(X)$ for the set of path-components of X and $[\mathbf{x}]$ for the path-component of a point $\mathbf{x} \in X$.

1.16 Example. $\pi_0(\mathbb{R} \setminus \{0\}) = \{(-\infty, 0), (0, \infty)\}$.

1.17 Proposition. Homeomorphic sets have the same number of path-components.

Proof. Suppose that X and Y are homeomorphic subsets of Euclidean spaces. Then there is a homeomorphism $f \colon X \to Y$. It it will be shown that this continuous function induces a bijection $f_* \colon \pi_0(X) \to \pi_0(Y)$ by $f_*([\mathbf{x}]) = [f(\mathbf{x})]$. This implies that $\pi_0(X)$ and $\pi_0(Y)$ have the same cardinality which is what we have prove.
The function f_* is well-defined because, if $[\mathbf{x}] = [\mathbf{x}']$ then $\mathbf{x} \sim \mathbf{x}'$ and so there is a path $\sigma \colon [0,1] \to X$ in X from $\mathbf{x}$ to $\mathbf{x}'$. It follows that $f \circ \sigma \colon [0,1] \to X \to Y$ is a path in Y from $f(\mathbf{x})$ to $f(\mathbf{x}')$ and so $f(\mathbf{x}) \sim f(\mathbf{x}')$, i.e. $[f(\mathbf{x})] = [f(\mathbf{x}')]$.
The function f_* is a bijection since it is easily checked that $(f^{-1})_* \colon \pi_0(Y) \to \pi_0(X)$, the function induced by the inverse $f^{-1} \colon Y \to X$, is an inverse for f_* (Exercise). $\square$ $\square$

Cut-points in subsets of Euclidean space

1.18 Definition. Suppose that X is a subset of some Euclidean space. Then a point $p \in X$ is called a *cut-point of type* n of X or an n*-point* of X if its complement $X \setminus \{p\}$ has n path-components.

1.19 Example. (a) In $[0,1)$ each $x \in (0,1)$ is a 2-point and 0 is a 1-point.

(b) In the subset of $\mathbb{R}^2$ given by the coordinate axes, $\{\, (x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0 \text{ or } x_2 = 0 \,\}$, $(0,0)$ is a 4-point whereas all other points are 2-points.

(c) In S^1 every point is a 1-point.

1.20 Theorem. Homeomorphic sets have the same number of cut-points of each type.

Proof. Let X and Y be homeomorphic subsets of Euclidean spaces. Then there is a homeomophism $f \colon X \to Y$. Suppose that $\mathbf{p} \in X$ is an n-point of X. Then f induces a homeomorphism $X \setminus \{\mathbf{p}\} \to Y \setminus \{f(\mathbf{p})\}$ and so these

subsets have the same number of path-components by Proposition 1.17. Hence $f(\mathbf{p})$ is an n-point of Y.
This shows that f induces a bijection between the n-points of X and the n-points of Y and so they must have the same number of n-points. $\square$

1.21 Example. $[0, 1)$ and S^1 are not homeomorphic since $[0, 1)$ has some 2-points (all of its points apart from 0) whereas S^1 has none.

Other applications of path-connectness

1.22 Theorem (The Brouwer Fixed Point Theorem in dimension 1). Suppose that $f\colon [-1, 1] \to [-1, 1]$ is a continuous map. Then f has a fixed point, i.e. there exists a point $t \in [-1, 1]$ such that $f(t) = t$.

Proof. Suppose for contradiction that f does not have a fixed point. Then $f(t) \neq t$ for all $t \in [-1, 1]$. Thus we may define a function $g\colon [-1, 1] \to \{-1, 1\}$ by $g(t) = (f(t) - t)/|f(t) - t|$. This is a continuous function from basic real analysis. However, since $f(-1) > -1$ and $f(1) < 1$ it follows that $g(-1) = 1$ and $g(1) = -1$. Hence g is a surjection. Hence, by Proposition 1.9, $\{-1, 1\}$ path-connected which contradicts the Intermediate Value Theorem (as in the proof of Proposition 1.11). Hence f has a fixed point. $\square$

1.23 Theorem (The Borsuk-Ulam Theorem in dimension 1). Suppose that $f\colon S^1 \to \mathbb{R}$ is a continuous function. Then there is a point $\mathbf{x} \in S^1$ such that $f(\mathbf{x}) = f(-\mathbf{x})$.

Proof. Exercise. Try a similar proof to that of Theorem 1.22. $\square$

1.24 Definition. A subset $A \subset \mathbb{R}^n$ is *bounded* if there is a real number R such that $\mathbf{x} \in A \implies |\mathbf{x}| \leqslant R$.

1.24 Theorem (The Pancake Theorem). Let A and B be bounded subsets of $\mathbb{R}^2$. Then there is a (straight) line in $\mathbb{R}^2$ which divides each of A and B in half by area.

Remark. The statement of this result assumes that A and B each have a well-defined area. In this course we ignore the technical difficulties associated with defining the area of a subset of $\mathbb{R}^2$ (the subject of integration and measure theory).

Outline Proof. Since A and B are bounded there is a real number R such that $\mathbf{a} \in A \Rightarrow |\mathbf{a}| \leqslant R$ and $\mathbf{x} \in B \Rightarrow |\mathbf{x}| \leqslant R$.

Suppose that $\mathbf{x} \in S^1$. For $t \in [-R, R]$ let $L_{\mathbf{x},t}$ denote the straight line through $t\mathbf{x}$ perpendicular to $\mathbf{x}$. Let $v(t) \in [0, 1]$ be the proportion of the area of A on the same side of $L_{\mathbf{x},t}$ as $R\mathbf{x}$. Then $v\colon [-R, R] \to [0, 1]$ is a continuous decreasing function with $v(-R) = 1$ and $v(R) = 0$. By the Intermediate Value Theorem there exists $t \in [-R, R]$ such that $v(t) = 1/2$. This t may not be unique but it is not difficult to show that $v^{-1}(1/2) = \{\, t \mid v(t) = 1/2 \,\} = [\alpha, \beta]$, a closed interval. Let $f_A(\mathbf{x}) = (\alpha + \beta)/2$. Then the line $L_{\mathbf{x},f_A(\mathbf{x})}$ bisects A.
The function $f_A\colon S^1 \to \mathbb{R}$ can be shown to be continuous. Furthermore $f_A(-\mathbf{x}) = -f_A(\mathbf{x})$ (since $L_{\mathbf{x},f_A(\mathbf{x})}$ and $L_{\mathbf{x},f_A(-\mathbf{x})}$ are the same line so that $f_A(\mathbf{x})\mathbf{x} = f_A(-\mathbf{x})(-\mathbf{x})$).
Similarly, using the region B, we may define a continuous function $f_B\colon S^1 \to \mathbb{R}$ such that $f_B(-\mathbf{x}). = -f_B(\mathbf{x})$ and $L_{\mathbf{x},f_B(\mathbf{x})}$ bisects B.
Let the continuous function $f\colon S^1 \to \mathbb{R}$ be given by $f(\mathbf{x}) = f_A(\mathbf{x}) - f_B(\mathbf{x})$. By the Borsuk-Ulam Theorem, there exists $\mathbf{x}_0 \in S^1$ such that $f(\mathbf{x}_0) = f(-\mathbf{x}_0)$. But $f(-\mathbf{x}_0) = f_A(-\mathbf{x}_0) - f_B(-\mathbf{x}_0) = -f_A(\mathbf{x}_0) + f_B(\mathbf{x}_0) = -f(\mathbf{x}_0)$. Hence $f(\mathbf{x}_0) = -f(\mathbf{x}_0)$ so that $f(\mathbf{x}_0) = 0$. This means that $f_A(\mathbf{x}_0) - f_B(\mathbf{x}_0) = 0$ so that $f_A(\mathbf{x}_0) = f_B(\mathbf{x}_0)$.
From the definition of f_A and f_B it follows that the line $L_{\mathbf{x}_0,f_A(\mathbf{x}_0)} = L_{\mathbf{x}_0,f_B(\mathbf{x}_0)}$ bisects both of A and B and so is the line whose existence is the claim of the theorem. $\square$