2 Topological Spaces

2.1 Problem. What properties of a subset of Euclidean space are preserved by a homeomorphism or topological equivalence, in other words determine 'the topology'?

2.2 Definition. Suppose X is a subset of a Euclidean space, $\mathbf{x}_0 \in X$ and $\varepsilon > 0$. Then the *open ε-ball about* $\mathbf{x}_0$ *(in X)* is the set

$$B^X_\varepsilon(\mathbf{x}_0) = \{\, \mathbf{x} \in X \mid |\mathbf{x} - \mathbf{x}_0| < \varepsilon \,\}.$$

For $X = \mathbb{R}^n$, this is often written $B_\varepsilon(\mathbf{x}_0)$ so that, for $X \subset \mathbb{R}^n$, $B^X_\varepsilon(\mathbf{x}_0) = B_\varepsilon(\mathbf{x}_0) \cap X$.

2.3 Remark. $B^X_\varepsilon(\mathbf{x}_0)$ in general depends on X, e.g. $B^{\mathbb{R}}_1(0) = (-1, 1)$, $B^{[0,\infty)}_1(0) = [0, 1)$, $B^{\mathbb{Z}}_1(0) = \{0\}$.

2.4 Proposition. A function $f\colon X \to Y$ of subsets of Euclidean spaces is continuous at $\mathbf{x}_0 \in X$ if and only if, for each real number $\varepsilon > 0$, there exists real number $\delta > 0$ such that

$$\mathbf{x} \in B^X_\delta(\mathbf{x}_0) \Rightarrow f(\mathbf{x}) \in B^Y_\varepsilon\big(f(\mathbf{x}_0)\big) \tag{1}$$

or (equivalently)

$$B^X_\delta(\mathbf{x}_0) \subset f^{-1}\Big(B^Y_\varepsilon\big(f(\mathbf{x}_0)\big)\Big). \tag{2}$$

Proof. (1) is a restatement of Definition 0.21 and then (2) is a rewrite of (1) using Definition 0.13(b). □

2.5 Definition. Suppose that X is a subset of a Euclidean space. A subset U of X is a *neighbourhood* of a point $\mathbf{x}_0 \in X$ if there exists a real number $\varepsilon > 0$ such that $B^X_\varepsilon(\mathbf{x}_0) \subset U$.
A subset $U \subset X$ is *open (in X)* if it is a neighbourhood of each point $\mathbf{x}_0 \in U$.

2.6 Example. (a) For any subset X of a Euclidean space, X is open in X using any $\varepsilon > 0$ since by definition $B^X_\varepsilon(\mathbf{x}_0) \subset X$.

(b) For any subset X of a Euclidean space, the empty set $\emptyset$ is open in X since the condition is vacuous.

(c) An open interval (a, b) is open in $\mathbb{R}$ [which means that the language is consistent] since for $x_0 \in (a, b)$ we can put $\varepsilon = \min(x_0 - a, b - x_0)$.

(d) The singleton set $\{0\}$ is not open in $\mathbb{R}$ (since $B_\varepsilon(0)$ contains $\varepsilon/2 \notin \{0\}$ for all $\varepsilon > 0$) but it is open in $\mathbb{Z}$ (since, taking $\varepsilon = 1$, $B_1^{\mathbb{Z}}(0) = \{0\} \subset \{0\}$).

2.7 Proposition. For $\mathbf{x}_0 \in X \subset \mathbb{R}^n$ and $\varepsilon > 0$, the open ε-ball $B_\varepsilon^X(\mathbf{x}_0)$ is open in X.

Proof. Given $\mathbf{x}_1 \in B_\varepsilon^X(\mathbf{x}_0)$, then $|\mathbf{x}_0 - \mathbf{x}_1| < \varepsilon$. Put $\varepsilon_1 = \varepsilon - |\mathbf{x}_0 - \mathbf{x}_1| > 0$. Then $B_{\varepsilon_1}^X(\mathbf{x}_1) \subset B_\varepsilon^X(\mathbf{x}_0)$ as required to prove that $B_\varepsilon(\mathbf{x})$ is open [for $\mathbf{x} \in B_{\varepsilon_1}^X(\mathbf{x}_1) \Rightarrow |\mathbf{x} - \mathbf{x}_1| < \varepsilon_1 = \varepsilon - |\mathbf{x}_0 - \mathbf{x}_1| \Rightarrow |\mathbf{x} - \mathbf{x}_1| + |\mathbf{x}_1 - \mathbf{x}_0| < \varepsilon \Rightarrow |\mathbf{x} - \mathbf{x}_0| < \varepsilon$ (since $|\mathbf{x} - \mathbf{x}_0| \leqslant |\mathbf{x} - \mathbf{x}_1| + |\mathbf{x}_1 - \mathbf{x}_0|$ by the Triangle Inequality) $\Rightarrow \mathbf{x} \in B_\varepsilon^X(\mathbf{x}_0)$.] $\square$

2.8 Theorem. A function $f\colon X \to Y$ of subsets of Euclidean spaces is continuous if and only if, for each open subset V of Y, the subset $f^{-1}(V)$ is open in X.

Proof. '$\Rightarrow$': Suppose that $f\colon X \to Y$ is continuous. Let V be an open subset of Y. Then, to see that $f^{-1}(V)$ is open in X, let $\mathbf{x}_0 \inf^{-1}(V)$. Then $f(\mathbf{x}_0) \in V$ and so, since V is open in Y, there exists $\varepsilon > 0$ such that $B_\varepsilon^Y\big(f(\mathbf{x}_0)\big) \subset V$. So $f^{-1}\big(B_\varepsilon^Y(f(\mathbf{x}_0))\big) \subset f^{-1}(V)$. Now, since f is continuous at $\mathbf{x}_0$, by Proposition 2.4 there exists $\delta > 0$ such that $B_\delta^X(\mathbf{x}_0) \subset f^{-1}\big(B_\varepsilon^Y(f(\mathbf{x}_0))\big)$. Thus $B_\delta^X(\mathbf{x}_0) \subset f^{-1}(V)$ as required to prove that $f^{-1}(V)$ is a neighbourhood of $\mathbf{x}_0$. Hence $f^{-1}(V)$ is open in X.
'$\Leftarrow$': Suppose that the condition in the Theorem holds and $\mathbf{x}_0 \in X$. To see that f is continuous at $\mathbf{x}_0$ suppose that $\varepsilon > 0$. Then, by Proposition 2.7, $B_\varepsilon^Y\big(f(\mathbf{x}_0)\big)$ is open in Y. Hence, by hypothesis, $f^{-1}\big(B_\varepsilon^Y(f(\mathbf{x}_0))\big)$ is open in X. So, since $\mathbf{x}_0 \in f^{-1}\big(B_\varepsilon^Y(f(\mathbf{x}_0))\big)$, there exists $\delta > 0$ such that $B_\delta^X(\mathbf{x}_0) \subset f^{-1}\big(B_\varepsilon^Y(f(\mathbf{x}_0))\big)$ and so, by Proposition 2.4, f is continuous at $\mathbf{x}_0$. Hence $f\colon X \to Y$ is continuous. $\square$

2.9 Corollary. A bijection $f\colon X \to Y$ of subsets of Euclidean spaces is a homeomorphism if and only if

$$U \text{ open in } X \Leftrightarrow f(U) \text{ open in } Y.$$

Proof. Suppose that $f\colon X \to Y$ is a bijection of subsets of Euclidean space. (a) For each $V \subset Y$, $V = f(U)$ where $U = f^{-1}(V)$ (since f is a bijection). Hence f is continuous if and only if (V open in $Y \Rightarrow f^{-1}(V)$ open in X) (Theorem 2.8) if and only if ($f(U)$ open in $Y \Rightarrow U$ open in X).

(b) Let $\mathbf{g} = f^{-1}\colon Y \to X$. Then $\mathbf{g}^{-1} = f$. Hence, for $U \subset X$, $\mathbf{g}^{-1}(U) = f(U)$. So $\mathbf{g} = f^{-1}$ is continuous if and only if (U open in $X \Rightarrow \mathbf{g}^{-1}(U)$ open in Y) if and only if (U open in $X \Rightarrow f(U)$ open in Y). $\square$

2.10 Remark. This result indicates that the answer to Problem 2.1 is that a homeomorphism is a bijection which preserves the open sets and so 'the topology' is determined by the open sets. So we can specify a 'topology' on any set X by declaring which subsets of X are to be the open sets.

2.11 Definition. Given a set X, a *topology* on X is a collection τ of subsets of X with the following properties:

(i) $\emptyset \in \tau$, $X \in \tau$;

(ii) the intersection of any two subsets in τ is in τ:

$$U_1, U_2 \in \tau \Rightarrow U_1 \cap U_2 \in \tau;$$

(iii) the union of any collection of subsets in τ is in τ:

$$U_\lambda \in \tau \text{ for all } \lambda \in \Lambda \Rightarrow \bigcup_{\lambda \in \Lambda} U_\lambda \in \tau.$$

A pair (X, τ) such that τ is a topology on X is called a *topological space*, usually denoted X when the topology is clear. The subsets in τ are called the *open subsets* of X (with the topology τ) or the *open sets* of the topology τ. Thus, given a topology τ on a set X the statements '$U \in \tau$' and 'U is a open subset in X' have precisely the same meaning.

2.12 Definition. Suppose that (X, τ_1) and (Y, τ_2) are topological spaces. A function $f\colon X \to Y$ is *continuous* (with respect to τ_1 and τ_2) if, for each open subset V of Y (with the topology τ_2), the subset $f^{-1}(V)$ is open in X (with the topology τ_1), i.e.

$$V \in \tau_2 \Rightarrow f^{-1}(V) \in \tau_1.$$

The function $f\colon X \to Y$ is a *homeomorphism* if it is a continuous bijection with continuous inverse.

2.13 Proposition. If $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous functions of topological spaces, then $g \circ f\colon X \to Z$ is a continuous function.

Proof. Exercise. $\square$

2.14 Proposition. A bijection $f\colon X \to Y$ between topological spaces is a homeomorphism if and only if

$$U \text{ is open in } X \Leftrightarrow f(U) \text{ is open in } Y.$$

Proof. Exercise. (This is identical to the proof of Corollary 2.9 with Definition 2.12 playing the role of Theorem 2.8.) □

2.15 Remark. The original definition of topological space given by Felix Hausdorff in 1914 was equivalent to Definition 2.11 with the addition of a fourth condition on the set of open sets (now called the Hausdorff condition to be discussed in §4).

2.16 Proposition. (a) Suppose that X is a subset of Euclidean space. Then the open subsets of X defined by Definition 2.5 are the open subsets of a topology on X (called *the usual topology* on X).

(b) A function $f\colon X \to Y$ of subsets of Euclidean spaces is continuous according to Definition 0.21 if and only if it is continuous according to Definition 2.12 with respect to the usual topologies on X and Y.

Proof. (a) To see that the open sets of a subset of Euclidean space X given by Defintion 2.5 define a topology on X we need to check the conditions in Definition 2.11.
(i) $\emptyset$ and X are open in X by Examples 2.6(a) and (b).
(ii) Suppose that $U_1 \subset X$ and $U_2 \subset X$ are open in X. Then $U_1 \cap U_2$ is open in X (Exercise).
(iii) Suppose that $U_\lambda \subset X$ is open in X for all $\lambda \in \Lambda$. Then $\bigcup_{\lambda\in\Lambda} U_\lambda$ is open in X (Exercise).
Hence these open sets are the open sets of a topology on X.
(b) This is a restatement of Theorem 2.8. □

2.17 Example. (a) The *discrete topology* on a set X consists of all the subsets of X.

If X has the discrete topology and Y is any topological space, then all functions $f\colon X \to Y$ are continuous.

(b) The *indiscrete topology* on a set X is given by $\tau = \{\emptyset, X\}$.

If X has the indiscrete topology and Y is any topological space, then all functions $f\colon Y \to X$ are continuous.

(c) Suppose that $X = \{a, b\}$, a two point set. Then there are four topologies on X:

(i) $\tau_1 = \{\emptyset, X\}$ (the indiscrete topology);

(ii) $\tau_2 = \{\emptyset, \{a\}, X\}$;

(iii) $\tau_3 = \{\emptyset, \{b\}, X\}$;

(iv) $\tau_2 = \{\emptyset, \{a\}, \{b\}, X\}$ (the discrete topology).

The topological spaces (X, τ_2) and (X, τ_3) are homeomorphic with a homeomorphism given by $a \mapsto b$, $b \mapsto a$. Either of these topologies on a set of two elements is called the *Sierpinski topology.*

(d) Suppose that $X = \{a, b, c\}$ is a three point set.

(i) $\{\emptyset, \{a, b\}, \{b, c\}, X\}$ is not a topology (intersection fails).

(ii) $\{\emptyset, \{a\}, \{b\}, X\}$ is not a topology (union fails).

(e) The *identity function* $\mathrm{id}_X \colon X \to X$, given by $\mathrm{id}_X(x) = x$ for all $x \in X$, is continuous with respect to any topology on the set X (Exercise).

(f) Given sets X and Y and a point $a \in Y$, the *constant function* $c_a \colon X \to Y$, given by $c_a(x) = a$ for all $x \in X$, is continous with respect to any topologies on X and Y (Exercise).

2.18 Remark. (a) By induction, the intersection of any *finite* number of open subsets in a topology is open. However, we do not require the intersection every collection of open subsets to be open. For example, in $\mathbb{R}$ with the usual topology, $\bigcap_{n=1}^{\infty}(-1/n, 1/n) = \{0\}$ which is not open.

(b) Continuity does not imply that the image of each open set if open. For example, in $\mathbb{R}$ with the usual topology the continuous map $f \colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ has $f(-1, 1) = [0, 1)$ which is not open.

A continuous function $f \colon X \to Y$ of topological spaces for which U open in X implies that $f(U)$ is open in Y is called an *open map.*

2.19 Definition. A subset A of a topological space X is *closed* when its complement $X \setminus A$ is open.

2.20 Proposition. In a topological space X,

(i) $\emptyset$ and X are closed;

(ii) the union of any pair of closed subsets is closed;

(iii) the intersection of any collection of closed subsets is closed.

Furthermore, any collection of subsets of X satisfying these conditions is the set of closed subsets of a topology on X.

Proof. This is immediate from Definition 2.11 using the set theoretic properties of complements of unions and intersections (the de Morgan Laws, Proposition 0.5(iv)). $\square$

2.21 Remark. The word 'closed' is not the same as 'not open'. For example, in $\mathbb{R}$ with the usual topology, $[0,1)$ is neither open nor closed whereas $\emptyset$ is both open and closed. In general, most subsets are neither open nor closed. Some subsets are both open and closed.

2.22 Definition. Suppose that X is a topological space. Then a collection $\mathcal{B}$ of open subsets in X is called a *basis* for the topology on X if every non-empty open subset in X can be expressed as a union of open subsets in $\mathcal{B}$.

2.23 Proposition. A basis for the usual topology on $\mathbb{R}$ is given by

$$\mathcal{B} = \{\, (a,b) \mid a, b \in \mathbb{R},\, a < b \,\}.$$

Proof. Given an open set $U \subset \mathbb{R}$ (in the usual topology) then, for each $x \in U$ there exists $\varepsilon_x > 0$ such that $B_{\varepsilon_x}(x) = (x - \varepsilon_x, x + \varepsilon_x) \subset U$. Then $U = \bigcup_{x \in U}(x - \varepsilon_x, x + \varepsilon_x)$ (the empty union if $U = \emptyset$). $\square$

2.24 Remark. The material in §1 about paths, path-connectedness and cut-points extends from subsets of Euclidean spaces to all topological spaces.

2.25 Definition. For a subset $A \subset X$ of a topological space X the *interior* A° is defined to be the union of all open subsets U of X, which are contained in A.
The *closure* $\overline{A}$ is defined as the intersection of all closed subsets of X which contain A.
The *boundary* ∂A is defined as the difference $\overline{A} \setminus A^\circ$.

2.26 Definition. An *open neighbourhood* of a point x in a topological space X is an open subset which contains x.

2.27 Exercise. A point x is contained in the interior of $A \subset X$ if and only if there exists an open neigbourhood U of x, which is contained in A.
A point $x \in X$ is contained in $\overline{A}$ if and only if every open neighbourhood of x intersects A.
A point $x \in X$ is contained in ∂A if and only if every open neighbourhood of x intersects A and its complement $X \setminus A$.

2.28 Example. The consider the interval $J = [0, 1)$ as a subset of $\mathbb{R}$ with the usual topology. Then we have $J^\circ = (0, 1)$ and $\overline{J} = [0, 1]$ and $\partial J = \{0, 1\}$. Indeed, $(0, 1)$ is an open subset, which is contained in J. Hence, $(0, 1) \subset J^\circ$, but every open subset which contains 0, will also contain the interval $(-\epsilon, \epsilon) \not\subset J$ for $\epsilon > 0$ sufficiently small. Hence, 0 is not cointained in the interior.
Similarly $[0, 1]$ is a closed subset, which contains J. Hence, $[0, 1) \subset \overline{J} \subset [0, 1]$. Hence, $\overline{J}$ has to be either $[0, 1)$ or $[0, 1]$. Since the closure of a set is a closed by definition. it follows, that $\overline{J} = [0, 1]$.
Now, we obtain $\partial J = [0, 1] \setminus (0, 1) = \{0, 1\}$.