3 Constructing New Spaces

Subspaces

3.1 Remark. Given a topological space X, is there a natural way of putting a topology on a subset $X_1 \subset X$? One desirable property is the following since we do not expect the codomain of a map to affect whether or not it is continuous.

For all topological spaces Y,
$f\colon Y \to X_1$ is continuous $\quad\Leftrightarrow\quad$ $i \circ f\colon Y \to X$ is continous.

Here $i\colon X_1 \to X$ denotes the inclusion map $i(x) = x$ for all $x \in X_1$.

There is just one topology on X_1 which has this property and this is known as the *subspace topology.* This property is known as the *universal property* of the subspace topology.

3.2 Definition. Given a topological space (X, τ) and a subset $X_1 \subset X$, then the *subspace topology* on X_1 (induced by τ) is given by $\tau_1 = \{\, U \cap X_1 \mid U \in \tau \,\}$, i.e. $V \subset X_1$ is open in X_1 if and only if $V = U \cap X_1$ where U is some open set in X.

With this topology we say that X_1 is a *subspace* of X.

3.3 Proposition. Given a topological space X and a subset $X_1 \subset X$, Definition 3.2 defines a topology on X_1. With this topology,

(a) the inclusion map $i\colon X_1 \to X$ is continuous;

(b) given a continuous function $f\colon X \to Y$ (where Y is an topological space), the restriction $f|X_1 = f \circ i\colon X_1 \to Y$ is continuous;

(c) (*the universal property*) a function $f\colon Y \to X_1$ (where Y is any topological space) is continuous if and only if $i \circ f\colon Y \to X$ is continuous.

Proof. To see that Definition 3.2 defines a topology we check the properties in Definition 2.11.

(i) $\emptyset$ and X are open in X and so $\emptyset \cap X_1 = X$ and $X \cap X_1 = X_1$ are open in X_1.

(ii) Given V_1 and V_2 open in X_1 then $V_i = U_i \cap X$ for U_i open in X ($i = 1, 2$). Hence $V_1 \cap V_2 = (U_1 \cap X_1) \cap (U_2 \cap X_1) = (U_1 \cap U_2) \cap X_1$ is open in X_1 since $U_1 \cap U_2$ is open in X.

(iii) Given V_λ open in X_1 for $\lambda \in \Lambda$. Then $V_\lambda = U_\lambda \cap X_1$ where U_λ is open in X ($\lambda \in \Lambda$). Hence $\bigcup_{\lambda\in\Lambda} V_\lambda = \bigcup_{\lambda\in\Lambda}(U_\lambda \cap X_1) = \left(\bigcup_{\lambda\in\Lambda} U_\lambda\right) \cap X$ is open in X_1 since $\bigcup_{\lambda\in\Lambda} U_\lambda$ is open in X.

(a) Given U open in X then $i^{-1}(U) = U \cap X_1$ is open in X_1 and so i is continuous.

(b) This follows from the fact the composition of continuous functions is continuous (Proposition 2.13).

(c) '$\Rightarrow$': This follows from the fact that the composition of continuous functions is continuous.

'$\Leftarrow$': Suppose that $f\colon Y \to X_1$ is a function from a topological space Y such that $i \circ f\colon Y \to X$ is continuous. Then given V open in X_1, $V = U \cap X_1 = i^{-1}(U)$ for U open in X. Thus $f^{-1}(V) = f^{-1}i^{-1}(U) = (i \circ f)^{-1}(U)$ is open in Y since $i \circ f$ is continuous. Hence f is continuous.

□

3.4 Remark. (a) The subspace topology on $X \subset \mathbb{R}^n$ induced by the usual topology on $\mathbb{R}^n$ is the usual topology on X. [Exercise. Note that $B_\varepsilon^X(x) = B_\varepsilon(x) \cap X$ for $x \in X$ and $\varepsilon > 0$.]

(b) Given a subspace X_1 of a topological space X it is not in general true that an open [closed] subset of X_1 is open [closed] in X. For example, $(1/2, 1]$ is open in $[0, 1]$ with the usual topology (since $(1/2, 1] = (1/2, 3/2) \cap [0, 1]$) but is not open in $\mathbb{R}$.

3.5 Proposition. Given a subspace X_1 of a topological space X, a subset $B \subset X_1$ is closed in X_1 if and only if $B = A \cap X_1$ where A is some closed set in X.

Proof. Exercise. □

3.6 Proposition. Suppose that X_1 is a subspace of a topological space X. Then all closed subsets of the subspace $X_1 \subset X$ are closed in X if and only if X_1 is a closed subset of X.

Proof. '$\Rightarrow$': If all closed subsets of X_1 are closed in X then, since X_1 is closed in X_1, it is closed in X.

'$\Leftarrow$': Suppose that X_1 is a closed subset of X. Then, given B closed in X_1, $B = A \cap X_1$ where A is closed in X (by Proposition 3.5) and so B is closed in X (the intersection of two closed subsets). $\square$

3.7 Theorem (Gluing Lemma). Suppose that X_1 and X_2 are closed subspaces of a topological space X such that $X = X_1 \cup X_2$. Suppose that $f_1 \colon X_1 \to Y$ and $f_2 \colon X_2 \to Y$ are continuous functions to a topological space Y such that, for all $x \in X_1 \cap X_2$, $f_1(x) = f_2(x)$. Then the function $f \colon X \to Y$ defined by $f(x) = f_1(x)$ if $x \in X_1$, $f(x) = f_2(x)$ if $x \in X_2$ is well-defined and continuous.

Proof. f is well-defined by the condition on f_1 and f_2 in the theorem. To see that f is continuous it is sufficient, by Problems 2, Question 8, to prove that the inverse image of a closed set in Y is closed in X. Given A closed in Y, $f_j^{-1}(A)$ is closed in X_j ($j = 1$,2) since f_j is continuous and so, using Proposition 3.6, $f_j^{-1}(A)$ is closed in X since X_j is closed in X. If follows that $f^{-1}(A) = f_1^{-1}(A) \cup f_2^{-1}(A)$ is closed in X and so f is continuous. $\square$

3.8 Example. This result gives a justification for the continuity of the product of two paths $\sigma_1 * \sigma_2$ in a topological space X (generalizing Definition 1.13(c) to topological spaces). For suppose that $\sigma_1 \colon [0,1] \to X$ and $\sigma_2 \colon [0,1] \to X$ are two paths in X so that $\sigma_1(1) = \sigma_2(0)$. Then the product path $\sigma_1 * \sigma_2 \colon [0,1] \to X$ is given by

$$\sigma_1 * \sigma_2(s) = \begin{cases} \sigma_1(2s) & \text{for } 0 \leqslant s \leqslant 1/2, \\ \sigma_2(2s-1) & \text{for } 1/2 \leqslant s \leqslant 1. \end{cases}$$

(generalizing Definition 1.13(c)). Define

$$f_1 \colon [0,1/2] \mapsto X \text{ to be the composition } [0,1/2] \xrightarrow{s \mapsto 2s} [0,1] \xrightarrow{\sigma_1} X, \text{ and}$$

$$f_2 \colon [1/2,1] \to X \text{ to be the composition } [1/2,1] \xrightarrow{s \mapsto 2s-1} [0,1] \xrightarrow{\sigma_2} X.$$

Then f_1 and f_2 are compositions of continuous functions and so continuous. We can apply the Gluing Lemma to these two functions since $[0,1/2]$ and $[1/2,1]$ are closed in $[0,1]$, the intersection $[0/1/2] \cap [1/2,1] = \{1/2\}$ and $f_1(1/2) = \sigma_1(1) = \sigma_2(0) = f_2(1/2)$. The well-defined continuous function given by the Lemma is $\sigma_1 * \sigma_2$.

Product spaces

3.9 Remark. Given topological spaces X_1 and X_2, is there a natural way of putting a topology on the cartesian product $X_1 \times X_2$? One desirable property is the following since it would generalize the familiar property that a function into $\mathbb{R}^n$ is continuous if and only if the coordinate functions are continuous (see Remarks 0.22(b)).

$$\boxed{\begin{gathered}\text{For all topological spaces } Y,\\ f\colon Y \to X_1 \times X_2 \text{ is continuous}\\ \Leftrightarrow\\ p_i \circ f\colon Y \to X_i \text{ is continous for } i = 1, 2.\end{gathered}}$$

Here $p_i\colon X_1 \times X_2 \to X_i$ denotes the projection map $p_i(x_1, x_2) = x_i$ for all $(x_1, x_2) \in X_1 \times X_2$.

There is just one topology on $X_1 \times X_2$ which has this property and this is known as the *product topology.* This property is known as the *universal property* of the product topology.

3.10 Definition. Given topological spaces X_1 and X_2. The *product topology* on $X_1 \times X_2$ is the topology with a basis $\{ U_1 \times U_2 \mid U_i \text{ open in } X_i \text{ for } i = 1, 2 \}$. With this topology $X_1 \times X_2$ is called the *product* of the spaces X_1 and X_2.

3.11 Proposition. Given topological spaces X_1 and X_2, the set given above is the basis for a topology on $X_1 \times X_2$. With this topology,

(a) the projection functions $p_i\colon X_1 \times X_2 \to X_i$ are continuous;

(b) (*the universal property*) a function $f\colon Y \to X_1 \times X_2$ (for Y any topological space) is continuous if and only if the coordinate functions $p_i \circ f\colon Y \to X_i$ are continuous for $i = 1, 2$.

Proof. To see that the collection of subsets in Definition 3.10 is a basis for a topology on $X_1 \times X_2$ we use the result of Problems 2, Question 11. Given two basic open sets $U_1 \times U_2$ and $U_1' \times U_2'$ in $X_1 \times X_2$ (i.e. U_i and U_i' are open in X_i for $i = 1, 2$), then

$$(U_1 \times U_2) \cap (U_1' \times U_2') = (U_1 \cap U_1') \times (U_2 \cap U_2')$$

(by Proposition 0.7(iii)) which is also a basic open set since $U_i \cap U_i'$ is open in X_i for $i = 1, 2$.

(a) For U open in X_1, $p_1^{-1}(U) = U \times X_2$ which is open in $X_1 \times X_2$. Hence p_1 is continuous. Similarly, p_2 is continuous.

(b) '$\Rightarrow$': This follows from the continuity of a composition of continuous functions.

'$\Leftarrow$': To prove that $f\colon Y \to X_1 \times X_2$ is continuous it is sufficient to prove that $f^{-1}(U_1 \times U_2)$ is open in Y for basic open sets $U_1 \times U_2$ by Problems 2, Question 9. Given such a basic open set and a function $f\colon Y \to X_1 \times X_2$ such that the coordinate functions $p_1 \circ f$ and $p_2 \circ f$ are continuous, $(p_1 \circ f)^{-1}(U_1) = f^{-1}p_1^{-1}(U_1) = f^{-1}(U_1 \times X_2)$ is open in Y and, similarly, $(p_2 \circ f)^{-1}(U_2) = f^{-1}(X_1 \times U_2)$ is open in Y. Hence, by taking the intersection of these open sets, $f^{-1}(U_1 \times X_2) \cap f^{-1}(X_1 \times U_2) = f^{-1}(U_1 \times X_2 \cap X_1 \times U_2) = f^{-1}(U_1 \times U_2)$ is open in Y and so f is continuous. $\square$

3.12 Remark. (a) In the same way we can define the product topology on any finite product $X_1 \times X_2 \times \cdots \times X_n$ of topological spaces: a basis is given by subsets of the form $U_1 \times U_2 \times \cdots \times U_n$ where U_i is an open subset of X_i.

(b) For each point $x_2 \in X_2$, the subspace $X_1 \times \{x_2\}$ of the product space $X_1 \times X_2$ is homeomorphic to X_1.

To see this we prove that the obvious bijection $f\colon X_1 \to X_1 \times \{x_2\}$ given by $f(x) = (x, x_2)$ for $x \in X_1$ is a homeomophism by using using the universal properties of the product topology and the subspace topology.

First of all, f is continuous if and only if $i_1 = i \circ f\colon X_1 \to X_1 \times x_2 \to X_1 \times X_2$ is continuous (by the universal property of the subspace topology) if and only if $p_1 \circ i_1 = I_{X_1}\colon X_1 \to X_1$ is continuous and $p_2 \circ i_1 = c_{x_2}\colon X_1 \to X_2$ is continuous (by the universal property of the product topology) and these maps are continuous by Examples 2.17(e) and (f). Hence f is continuous.

Secondly, the function $f^{-1}\colon X_1 \times \{x_2\} \to X_1$ is the restriction of the projection map $p_1\colon X_1 \times X_2 \to X_1$ which is continuous by Proposition 3.11(a) and so is continuous by Proposition 3.3(b).

(c) Given subsets $Y_1 \subset X_1$ and $Y_2 \subset X_2$ of topological spaces X_1 and X_2 then $Y_1 \times Y_2$ may be topologized as (i) a subspace of the product space $X_1 \times X_2$, and (ii) the product of the subspaces Y_1 and Y_2. These two topologies are the same. [Exercise. Use the universal properties to show that the identity map $I_{Y_1 \times Y_2}\colon (Y_1 \times Y_2, \tau_1) \to (Y_1 \times Y_2, \tau_2)$ is a homeomorphism where τ_1 is the topology (i) and τ_2 is the topology τ_2.]

3.13 Example. (a) Euclidean n-space $\mathbb{R}^n$ with the usual topology is homeomorphic to the product space $\mathbb{R}^{n-1} \times \mathbb{R}$ (with the usual topologies on $\mathbb{R}$ and $\mathbb{R}^{n-1}$. [Exercise.]

(b) If X and Y have the discrete topology then the product topology on $X \times Y$ is the discrete topology.

(c) The product space $[0,1] \times S^1$ is called the *cylinder*.

(d) The product space $S^1 \times S^1$ is called the *torus*.

(e) The product space $D^2 \times S^1$ is called the *solid torus*.