Spring Semester 2019–2020

MATH31052 *Topology*

Quotient spaces

3.14 Definition. Suppose that $q\colon X \to Y$ is a surjection from a topological space X to a set Y. Then the *quotient topology* (or the *identification topology*) on Y *determined by* q is given by the condition $V \subset Y$ is open in Y if and only if $q^{-1}(V)$ is open in X. With this topology we call Y a *quotient space* of X.

3.15 Proposition. Given a surjection $q\colon X \to Y$ from a topological space X to a set Y, the above definition gives a topology on Y. With this topology,

(a) the function $q\colon X \to Y$ is continuous;

(b) (*the universal property*) a function $f\colon Y \to Z$ to a topological space Z is continuous if and only if the composition $f \circ q\colon X \to Z$ is continuous.

Proof. Exercise. You might use the proof of Proposition 3.3 as a model. □

3.16 Remark. Given an equivalence relation $\sim$ on a topological space X there is a surjection $q\colon X \to X/{\sim}$ to the set of equivalence classes given by sending each element of X to its equivalence class: $q(x) = [x] = \{\, x' \in X \mid x' \sim x \,\}$ (see Definition 0.18). We can give $X/{\sim}$ the quotient topology determined by q. We call such a quotient space an *identification space of* X.

3.17 Definition. Suppose that A is a non-empty subset of a topological space X. Then we may define an equivalence relation on X by

$$x \sim x' \Leftrightarrow x = x' \text{ or both } x \text{ and } x' \in A.$$

In this case we write the set of equivalence classes $X/{\sim}$ as X/A. With the quotient topology this is called the identification space obtained from X by *collapsing A to a point.* Notice that, as a set, X/A has one point for each point $x \in X \setminus A$ (since $[x] = \{x\}$) and one point corresponding to all of A (since if $a \in A$, $[a] = A$).

3.18 Theorem. Suppose that $f\colon X \to Y$ is a continuous surjection of topological spaces. Then we may define an equivalence relation on X by

$$x \sim x' \Leftrightarrow f(x) = f(x')$$

and then the bijection $F\colon X/{\sim} \to Y$ induced by f, i.e. $F([x]) = f(x)$ for $x \in X$ (see Theorem 0.20), is a continuous bijection of topological spaces.

Proof. The proof that F is a bijection is the proof of Theorem 0.20.
The continuity of F follows from the universal property of the quotient topology since $F\circ q(x) = F\big(q(x)\big) = F([x]) = f(x)$ so that $F\circ q = f\colon X \to Y$ which is continuous. $\square$

3.19 Example. $[0,1]/\{0,1\} \cong S^1$.

The homeomorphism is induced by the continuous function $f\colon [0,1] \to S^1$ given by $f(t) = (\cos 2\pi t, \sin 2\pi t)$.

To see this notice that f is continuous (since the component functions are continuous) and

$$f(t) = f(t') \Leftrightarrow t = t' \text{ or } t, t' \in \{0,1\}. \tag{1}$$

Then, by Theorem 3.18, f induces a continuous bijection $F\colon [0,1]/\{0,1\} \to S^1$ by $F[t] = f(t)$. It will follow from a result in §5 that this is a homeomorphism but this may be shown directly as follows.
The inverse $F^{-1}\colon S^1 \to [0,1]/\{0,1\}$ is given by

$$F^{-1}(\mathbf{x}) = \begin{cases} q(\cos^{-1}(x_1)/2\pi) & \text{for } x_2 \geqslant 0, \\ q(1-\cos^{-1}(x_1)/2\pi) & \text{for } x_2 \leqslant 0, \end{cases}$$

writing $\mathbf{x} = (x_1, x_2)$.This uses the principal value of the inverse cosine, $\cos^{-1}\colon [-1,1] \to [0,\pi]$ which is a continuous function, and the continuous quotient map $q\colon [0,1] \to [0,1]/\{0,1\}$. This function is continuous on the two closed subsets $\{\,\mathbf{x} \in S^1 \mid x_2 \geqslant 0\,\}$ and $\{\,\mathbf{x} \in S^1 \mid x_2 \leqslant 0\,\}$, and agrees on their intersection $\{(\pm 1, 0)\}$. Hence, by the Gluing Lemma, $F^{-1}\colon S^1 \to [0,1]/\{0,1\}$ is continuous and so F is a homeomorphism.

3.20 Example. (a) There is a continuous bijection $D^n/S^{n-1} \to S^n$ (which is in fact a homeomorphism). Define $f\colon D^n \to S^n$ by

$$f(\mathbf{x}) = \left(\frac{2\sqrt{|\mathbf{x}| - |\mathbf{x}|^2}}{|\mathbf{x}|}\,\mathbf{x}, 1 - 2|\mathbf{x}|\right).$$

This is continuous since the component functions are continuous (actually this formula doesn't make sense if $\mathbf{x} = \mathbf{0}$ but, as $\mathbf{x} \to \mathbf{0}$, $f(\mathbf{x}) \to (\mathbf{0}, 1)$ and so if we put $f(\mathbf{0}) = (\mathbf{0}, 1)$ we get a continuous function). We can check that $|f(\mathbf{x})| = 1$ so that $f(\mathbf{x}) \in S^n$. Given $\mathbf{y} \in S^n$, $f(\mathbf{x}) = \mathbf{y} \Leftrightarrow 1 - 2|\mathbf{x}| = y_{n+1}$ and $2(\sqrt{|\mathbf{x}| - |\mathbf{x}|^2}/|\mathbf{x}|)\,\mathbf{x} = (y_1, \ldots, y_n)$. This means that for $\mathbf{y} \in S^n$ there is a unique $\mathbf{x} \in D^n$ such that $f(\mathbf{x}) = \mathbf{y}$ so long as $|y_{n+1}| < 1$. For $y_{n+1} = 1$, $f(\mathbf{x}) = (\mathbf{0}, 1) \Leftrightarrow \mathbf{x} = \mathbf{0}$.

For $y_{n+1} = -1$, $f(\mathbf{x}) = (\mathbf{0}, -1) \Leftrightarrow |\mathbf{x}| = 1$. Hence f is a continuous surjection and

$$f(\mathbf{x}) = f(\mathbf{x}') \Leftrightarrow \mathbf{x} = \mathbf{x}' \text{ or } \mathbf{x}, \mathbf{x}' \in S^{n-1}. \tag{2}$$

Define $F\colon D^n/S^{n-1} \to S^n$ by $F([\mathbf{x}]) = f(\mathbf{x})$. This is well-defined and an injection by (2). It is a surjection since f is a surjection. Thus F is a continuous bijection. It will follow from a result in §5 that F is a homeomorphism (a direct proof is a little awkward in this case).

(b) There is an equivalence relation on the unit square $I^2 = I \times I$ (where $I = [0, 1]$ with the usual topology) such that there is a continuous bijection $I^2/\sim \to I \times S^1$, the cylinder (which is in fact a homeomorphism). To see this, define a surjection $f\colon I^2 \to I \times S^1$ by $f(x, y) = \big(x, \exp(2\pi i y)\big)$ where we think of S^1 as $\{ z \in \mathbb{C} \mid |z| < 1 \}$ using the standard identification $\mathbb{C} \cong \mathbb{R}^2$. This function is continuous by the universal property of the product topology since the component functions are continuous and so Theorem 3.18 applies giving a continuous bijection $F\colon I^2/\sim \to I \times S^1$. It will follow from a result in §5 that this is a homeomorphism. It is possible to prove directly that F^{-1} is continuous using the Gluing Lemma using a argument like of that at the end of Example 3.19.

The equivalence relation on I^2 can be described explicitly by

$$(x, y) \sim (x', y') \Leftrightarrow \begin{cases} (x, y) = (x', y') \text{ or} \\ x = x',\ y = 0 \text{ and } y = 1 \text{ or} \\ x = x',\ y = 1 \text{ and } y' = 0. \end{cases}$$

We say that this equivalence relation is *generated by* the relation $(x, 0) \sim (x, 1)$ for all $x \in I$ (since the other relations are forced by reflexivity and symmetry).

(c) There is an equivalence relation on the unit square I^2 such that there is a continuous bijection $I^2/\sim \to S^1 \times S^1$ (and again this is in fact a homeomorphism).

This is left as an exercise. The equivalence relation is generated by $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, y)$ for all x, $y \in I$

(d) We may generate an equivalence relation on I^2 by $(x, 0) \sim (1 - x, 1)$ for all $x \in I$. The identification space $I^2/\sim$ is called the *Möbius band.*

(e) We may generate an equivalence relation on I^2 by $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, 1-y)$ for all x, $y \in I$. The identification space $I^2/\sim$ is called the *Klein bottle*.

(f) We may define an equivalence relation on I^2 by $(x, 0) \sim (1-x, 1)$ and $(0, y) \sim (1, 1-y)$ for all x, $y \in I$. The identification space $I^2/\sim$ is homeomorphic to a space called the *projective plane*, denoted P^2, usually defined as follows.

3.21 Definition. Define an equivalence relation on S^n by $\mathbf{x} \sim \pm\mathbf{x}$ for all $\mathbf{x} \in S^n$. Then the identification space $S^n/\sim$ is called *(real) projective n-space* and is denoted P^n (or sometimes $\mathbb{R}P^n$).

3.22 Remark. The formal proof that the identification space of Example 3.20(f) is homeomorphic to the projective plane P^2 is omitted.