4 Hausdorff Spaces

4.1 Definition. Suppose that (a_n) is a sequence of points in a topological space X and $a \in X$. Then $a_n \to a$ (as $n \to \infty$) when, for each open set U containing a, there exists an integer N such that

$$n \geqslant N \Rightarrow a_n \in U.$$

In this case we say that the sequence (a_n) *converges to* a or a is a *limit* of the sequence (a_n).

4.2 Example. (a) For $X = \mathbb{R}^n$ with the usual topology this is equivalent to the usual definition.

(b) For X a discrete space, $a_n \to a$ means that there is an integer N such that $n \geqslant N \Rightarrow a_n = a$ since $\{a\}$ is an open set and $a_n \in \{a\} \Rightarrow a_n = a$.

(c) For X an indiscrete space, every sequence converges to every point since the only open set containing $a \in X$ is X.

(d) For X with the Sierpinski topology $\{\emptyset, \{a\}, X\}$, $a_n \to a$ if and only if eventually $a_n = a$ (as in (b)), but $a_n \to b$ for all sequences (as in (c)).

4.3 Definition. The topological space X is *Hausdorff* (or T_2) if, for each distinct pair of points x, $y \in X$, there exist open sets U and V in X such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

4.4 Proposition. In a Hausdorff space a sequence can have at most one limit. So in this case we can refer to *the* limit of a convergent sequence and denote it by $\lim_{n\to\infty} a_n$.

Proof. Exercise (a proof by contradiction). □

4.5 Proposition. A subset $X \subset \mathbb{R}^n$ with the usual topology is Hausdorff.

Proof. Exercise. □

4.6 Proposition. Points are closed in a Hausdorff space, i.e. given $a \in X$, a Hausdorff space, the singleton subset $\{a\}$ is a closed subset.

Proof. Exercise. □

4.7 Remark. Hausdorff's original definition of a topological space (1914) was equivalent to our definition of a Hausdorff or T_2 space. A topological space in which singleton subsets are closed is called a *Fréchet space* or a T_1 space.

4.8 Proposition. (a) A subspace of a Hausdorff space is Hausdorff.

(b) The disjoint union of two Hausdorff spaces is Hausdorff.

(c) The product of two Hausdorff spaces is Hausdorff.

Proof. (a) and (b) Exercises.

(c) Suppose that X_1 and X_2 are Hausdorff spaces and that (x_1, x_2), $(x_1', x_2') \in X_1 \times X_2$ with $(x_1, x_2) \neq (x_1', x_2'$. Then $x_1 \neq x_1'$ or $x_2 \neq x_2'$.
If $x_1 \neq x_1'$, since X_1 is Hausdorff there are open subsets U, $V \subset X_1$ such that $x_1 \in U$, $x_2 \in V$ and $U \cap V = \emptyset$. Then $U \times X_2$ and $V \times X_2$ are disjoint open subsets of $X_1 \times X_2$ as required.

There is a similar argument if $x_2 \neq x_2'$.

So in either case (x_1, x_2) and (x_1', x_2') lie in disjoint open subsets of $X_1 \times X_2$ as required to prove that $X_1 \times X_2$ is Hausdorff. $\square$

4.9 Remark. A quotient space of a Hausdorff space is not necessarily Hausdorff. For example, define an equivalence relation on the closed interval $[-1, 1]$ with the usual topology (a Hausdorff space by Proposition 4.5) by $t \sim \pm t$ for $|t| < 1$. Then the quotient space $[-1, 1]/\!\sim$ is not Hausdorff.

Proof. Consider the points $q(-1) = [-1] = \{-1\}$ and $q(1) = [1] = \{1\} \in [-1, 1]/\!\sim$, writing $q\colon [-1, 1] \to [-1, 1]/\!\sim$ for the quotient map as usual.

Suppose for contradiction that there are disjoint open subsets U and $V \subset [-1, 1]/\!\sim$ such that $[-1] \in U$ and $[1] \in V$.

Since $[-1] \in U$, $q^{-1}(U) \subset [-1, 1]$ is an open set containing -1 and so, by the definition of an open set in the usual topology, since $-1 \in q^{-1}(U)$, there exists $\varepsilon_1 > 0$ such that $B_{\varepsilon_1}^{[-1,1]} = [-1, -1 + \varepsilon_1) \subset q^{-1}(U)$. This means that $(1 - \varepsilon_1, 1) \subset U$ since each point is equivalent to a point of $[-1, -1 + \varepsilon_1)$.

Similarly, since $[1] \in V$, there exists $\varepsilon_2 > 0$ such that $(1 - \varepsilon_2, 1] \subset q^{-1}(V)$. But then, for $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$, $1 - \varepsilon/2 \in (1 - \varepsilon, 1) = (1 - \varepsilon_1, 1) \cap (1 - \varepsilon_2, , 1] subset q^{-1}(U) \cap q^{-1}(V) = q^{-1}(U \cap V)$ so that $[1 - \varepsilon/2] \in U \cap V$ which are therefore not disjoint. So there do not exist disjoint open subsets of $[-1, 1]/\!\sim$ containing $[-1]$ and $[1]$. $\square$

4.10 Remark. $X = [-1,1]/\sim$ is covered by two open subsets, which are both homeomorphic to the unit interval $[0,1]$. Indeed, consider $U^- = X \setminus \{[1]\}$. Then $q^{-1}(U^-) = [-1,1)$ which is open in $[-1,1]$. On the other hand,

$$\varphi\colon [0,1] \to U^-;\ t \mapsto [-t]$$

gives a homeomorphism with inverse induced by

$$f\colon [-1,1) \to [0,1];\ t \mapsto |t|.$$

Indeed, one has $f(t) = f(s)$ if and only if $t \sim s$. Here, continuity follows by the continuity of $t \mapsto -t$, q, $t \mapsto |t|$ and by the universal property of the quotient topology *[Exercise]*. Similarly $U^+ = X \setminus \{[-1]\}$ is an open subset homeomorphic to $[0,1]$.