5 Compactness

5.1 Definition. A collection $\mathcal{F}$ of open subsets of a topological space X is called an *open cover* for a subset $A \subset X$ if $A \subset \bigcup_{U \in \mathcal{F}} U$. If $\mathcal{F}$ and $\mathcal{F}'$ are open covers for A and $\mathcal{F}' \subset \mathcal{F}$ then $\mathcal{F}'$ is a *subcover* of $\mathcal{F}$.
A subset K of a topological space X is *compact* if each open cover for K has a finite subcover. In particular, if X itself is a compact subset then we say that it is a *compact topological space*.

5.2 Example. (a) A finite subset $A = \{ a_1, a_2, \ldots, a_n \}$ of any topological space X is compact.

Proof. Given an open cover $\mathcal{F}$ for A we can choose, for each a_i, an open set $U_i \in \mathcal{F}$ such that $a_i \in U_i$. Then $\{ U_1, U_2, \ldots, U_n \}$ is the required finite subcover of $\mathcal{F}$ for A. Hence A is a compact subset. $\square$

(b) A subset of a discrete topological space is compact if and only if it is finite

Proof. '$\Leftarrow$': This is a particular case of 5.2(a).
'$\Rightarrow$': We prove the contrapositive. Suppose that $A \subset X$ is an infinite subset of a discrete topological space. Then $\mathcal{F} = \big\{ \{a\} \mid a \in A \big\}$ is an open cover for A with no finite subcover. Hence A is not compact. $\square$

(c) The subset $(0, 1)$ of $\mathbb{R}$ with the usual topology is not compact.

Proof. $\mathcal{F} = \{ (a, 1) \mid a \in (0, 1) \}$ is an open cover for $(0, 1)$ since, given $x \in (0, 1)$, $x \in (x/2, 1)$. This has no finite subcover for $(0, 1)$ since, given $\{ (a_1, 1), (a_2, 1), \ldots, (a_n, 1) \} \subset \mathcal{F}$, $\bigcup_{i=1}^{n} (a_i, 1) = (a, 1)$ where $a = \min\{a_i\}$ and $a/2 \notin (a, 1)$. $\square$

(d) $\mathbb{R}$ with the usual topology is not compact.

Proof. $\mathcal{F} = \{ (-n, n) \mid n \in \mathbb{N} \}$ is an open cover for $\mathbb{R}$ with no finite subcover since $\bigcup_{i=1}^{k} (-n_i, n_i) = (-n, n)$ where $n = \max\{n_i\}$ and $n + 1 \notin (-n, n)$. $\square$

(e) Given a non-compact topological space (X, τ) consider the set $X^* = X \sqcup \{\infty\}$ and the topology

$$\tau^* = \tau \cup \{X \setminus C \cup \{\infty\} \mid C \subset X \text{ compact}\}.$$

Then (X^*, τ^*) is a compact topological space (called the *one-point compactification*) [Excercise].

5.3 Proposition. Given a subset X_1 of a topological space X. The subspace X_1 is compact if and only if the subset X_1 is a compact subset of the topological space X.

Proof. Exercise. □

5.4 Proposition. Suppose that $f\colon X \to Y$ is a continuous function of topological spaces and K is a compact subset of X. Then $f(K)$ is a compact subset of Y.

Proof. Suppose that $\mathcal{F}$ is an open cover for $f(K)$. Let $f^{-1}(\mathcal{F}) = \{f^{-1}(V) \mid V \in \mathcal{F}\}$. Then $f^{-1}(\mathcal{F})$ is an over cover for K since, given $a \in K$, $f(a) \in f(K)$ so that $f(a) \in V$ for some $V \in \mathcal{F}$. Hence $a \in f^{-1}(V)$ for some $V \in \mathcal{F}$. Now, since K is compact, $f^{-1}(\mathcal{F})$ has a finite subcover for K,

$$\{\, f^{-1}(V_1), f^{-1}(V_2), \ldots, f^{-1}(V_n) \,\}.$$

. Thus, given $b \in f(K)$, $b = f(a)$ for some $a \in K$. Then $a \in f^{-1}(V_i)$ for some i, $1 \leqslant i \leqslant n$, so that $b = f(a) \in V_i$. Hence $\{\, V_1, V_2, \ldots, V_n \,\}$ is a finite subcover of $\mathcal{F}$ for $f(K)$.
Hence $f(K)$ is compact. □

5.5 Proposition. Compactness is a topological property, i.e. if X and Y are homeomorphic spaces then X is compact if and only if Y is compact.

Proof. Suppose that $X \cong Y$.
Suppose that X is compact. Then a homeomorphism is a continuous bijection $f\colon X \to Y$ and so, in particular, a continuous surjection. Hence $Y = f(X)$ is compact by Proposition 5.4.
Similarly, if Y is compact then X is compact. □

Compactness and accumulation points

5.6 Definition. Given a sequence (a_n) in a topological space X. Then $a \in X$ is called an *accumulation point* if every open neighbourhood $U \ni a$ contains infinitely many members of the sequence, i.e. $a_n \in U$ for infiniety many $n \in \mathbb{N}$.

5.7 Theorem. Every sequence in a compact topological space X has an accumulation point.

Proof. Assume there is no accumulation point. Then for every $x \in X$ there is open neighbourhood $U_x \ni x$, such that only finitely many sequence members are contained in U_x. Clearly $\{U_x \mid x \in X\}$ is an open cover of X. Now, compactness implies the existence of a finite subcover $\{U_{x_1}, \dots, U_{x_\ell}\}$. Note, that only for finitely many $n \in \mathbb{N}$ we have $a_n \in U_{x_i}$. Hence, ther exists an N_i such that $a_n \notin U_{x_i}$ for $n > N_i$. Then $a_n \notin \bigcup_{i=1}^{\ell} U_{x_i} = X$ for $n > N = \max\{N_1, \dots, N_\ell\}$, but this a contradiction, since (a_n) was a sequence in X, i.e. $a_n \in X$ for every $n \in \mathbb{N}$. $\square$

Compact Hausdorff spaces

5.8 Proposition. A compact subset of a Hausdorff space is closed.

Proof. Suppose that K is a compact subset of a Hausdorff space X. To prove that K is closed we prove that $X \setminus K$ is open, and we prove this by proving that it is a union of open subsets. Let $x \in X \setminus K$. Then, by the Hausdorff condition, for each $a \in K$ there are open subsets U_a, V_a of X such that $a \in U_a$, $x \in V_a$ and $U_a \cap V_a = \emptyset$.
Then $\{\, U_a \mid a \in A \,\}$ is an open cover for K. Hence, since K is compact, there is a finite subcover $\{U_{a_i} \mid 1 \leqslant i \leqslant n\}$ for K. So $K \subset \bigcup_{i=1}^{n} U_{a_i}$.
Put $V_x = \bigcap_{i=1}^{n} V_{a_i}$. Then V_x is a finite intersection of open sets and so is open and, since $x \in V_{a_i}$ for all i, $x \in V_x$. Furthermore, for $1 \leqslant i \leqslant n$, $V_x \cap U_{a_i} \subset V_{a_i} \cap U_{a_i} = \emptyset$ and so $V_x \cap U_{a_i} = \emptyset$. Hence $V_x \cap \bigcup_{i=1}^{n} U_{a_i} = \emptyset$ and so $X_x \cap K = \emptyset$ or, equivalently, $V_x \subset X \setminus K$.
Thus $X \setminus K = \bigcup_{x \in X \setminus K} V_x$ is a union of open sets and so is open. Hence K is closed. $\square$

5.9 Proposition. Suppose that K is a compact subset of a topological space X and A is a closed subset of X such that $A \subset K$. Then A is a compact subset of X.

Proof. Using the notation in the proposition, let $\mathcal{F}$ be an open cover for A. To prove that $\mathcal{F}$ has a subcover for A, observe that $\mathcal{F} \cup \{X \setminus A\}$

is an open cover for K, since A is closed, and so has a finite subcover $\{U_1, U_2, \ldots, U_n, X \setminus A\}$ where $U_i \in \mathcal{F}$ (we may as well assume $X \setminus A$ is one of the open sets in the subcover since we could add it if it wasn't included). Then $\{U_1, U_2, \ldots, U_n\}$ is a finite subcover of $\mathcal{F}$ for A as required to prove that A is compact. $\square$

5.10 Theorem. Suppose that $f\colon X \to Y$ is a continuous bijection from a compact space to a Hausdorff space. Then f is a homeomorphism.

Proof. Using the notation in the theorem, we prove that $f^{-1}\colon Y \to X$ is continuous by using closed subsets (see Problems 2, Question 8), i.e. we make use of the observation that $f^{-1}\colon Y \to X$ is continuous if and only if, when $A \subset X$ is a closed subset of X, $(f^{-1})^{-1}(A) = f(A)$ is a closed subset of Y. This condition holds from results we have already proved as follows.

A is a closed subset of compact X

$\Rightarrow$ A is a compact subset of X (by Proposition 5.9

$\Rightarrow$ $f(A)$ is a compact subset of Y (by Proposition 5.4)

$\Rightarrow$ $f(A)$ is a closed subset of Hausdorff Y (by Proposition 5.8).

Hence f^{-1} is continuous and so f is a homeomorphism. $\square$

Products of compact spaces

5.11 Theorem. The product $X_1 \times X_2$ of two non-empty topological spaces is compact if and only if the topological spaces X_1 and X_2 are both compact.

Proof. Suppose that X_1 and X_2 are non-empty topological spaces.

'$\Rightarrow$': Suppose that the product space $X_1 \times X_2$ is compact. Then, $p_1\colon X_1 \times X_2 \to X_1$, the projection map given by $p_1(x_1, x_2) = x_1$, is a surjection since X_2 is non-empty. Hence X_1 is compact since the continuous image of a compact space is compact (Proposition 5.4). Similarly, X_2 is compact. $\square$

To prove the converse the following lemma is useful.

5.12 Lemma. Let $\mathcal{B}$ be a basis for the topology of a topological space X. Then $K \subset X$ is compact if and only if every open cover for K by open sets in the basis $\mathcal{B}$ has a finite subcover.

Proof. '$\Rightarrow$': This is a special case of the definition.

'$\Leftarrow$': Suppose that K is a subset satisfying the condition regarding covers by basic open sets. Let $\mathcal{F}$ be an open cover for K. The we can write each open set of $\mathcal{F}$ as a union of basic open sets. Let $\mathcal{F}_1$ be the set of all basic open sets which are used in this process. Then $\bigcup_{V\in\mathcal{F}_1} V = \bigcup_{U\in\mathcal{F}} U$ and so $\mathcal{F}_1$ is an open cover for K by basic open sets. Hence, by the given condition, $\mathcal{F}_1$ has a finite subcover $\mathcal{F}_1'$ for K. For each basic open set V in $\mathcal{F}_1'$ we can choose an open set U in $\mathcal{F}$ which contains it as a subset so that $V \subset U$. This gives a finite subset $\mathcal{F}'$ of $\mathcal{F}$ such that $\bigcup_{U\in\mathcal{F}'} U \supset \bigcup_{V\in\mathcal{F}_1'} V \supset K$ and so $\mathcal{F}'$ is a finite subcover for K as required to prove that K is compact. □

Proof of Theorem 5.11 (continued): '$\Leftarrow$': Suppose that X_1 and X_2 are compact. Let $\mathcal{F}$ be an open cover for $X_1 \times X_2$ by basic open sets (i.e. sets of the form $U \times V$, where U is open in X_1 and V is open in X_2). Then, for $x \in X_2$, $\mathcal{F}$ is an open cover for $X_1 \times \{x\} \cong X_1$ (Remark 3.12(b)) which is compact. Hence $\mathcal{F}$ has a finite subcover

$$\mathcal{F}_x = \{\, U_1^x \times V_1^x, U_2^x \times V_2^x, \ldots, U_{n_x}^x \times V_{n_x}^x \,\}$$

for $X_1 \times \{x\}$, where each $U_i^x \times V_i^x$ is in $\mathcal{F}$ and $x \in V_i^x$ for each i.
Put $V_x = V_1^x \cap V_2^x \cap \cdots \cap V_{n_x}^x$ which is open (finite intersection of open sets) and non-empty since it contains x. Then $\mathcal{F}_x$ is an open cover for $X_1 \times V_x$.
Now $\{\, V_x \mid x \in X_2 \,\}$ is an open cover for X_2. Hence, since X_2 is compact, this has a finite subcover $\{\, V_{x_1}, V_{x_2}, \ldots, V_{x_m} \,\}$ for X_2. Then $\mathcal{F}_{x_1} \cup \mathcal{F}_{x_2} \cup \cdots \cup \mathcal{F}_{x_m}$ is a finite subcover of $\mathcal{F}$ for $X_1 \times X_2$.
Hence, by the lemma, $X_1 \times X_2$ is compact. □

Compact subsets of Euclidean spaces

5.13 Definition. A subset $A \subset \mathbb{R}^n$ is *bounded* if there is a real number M such that $|a| \leqslant M$ for all $a \in A$.

5.14 Theorem (Heine-Borel-Lebesgue Theorem). A subset of $\mathbb{R}^n$ with the usual topology is compact if and only if it is closed and bounded.

The proof depends on various results as follows.

5.15 Lemma. A compact subset of $\mathbb{R}^n$ with the usual topology is bounded.

Proof. Suppose that $X \subset \mathbb{R}^n$ is compact (usual topology). Then $\{B_n(\mathbf{0}) \mid n \in \mathbb{N}\}$ is an open cover for $\mathbb{R}^n$ and so an open cover for X. Hence, since X is compact, there is a finite subcover $\{\, B_{n_1}(\mathbf{0}), B_{n_2}(\mathbf{0}), \ldots, B_{n_k}(\mathbf{0}) \,\}$ for X. Let $n = \max\{n_i\}$. Then $X \subset \bigcup_{i=1}^k B_{n_i}(\mathbf{0}) = B_n(\mathbf{0})$ and so is bounded. □

5.16 Lemma. A compact subset of $\mathbb{R}^n$ is closed.

Proof. This follows from Proposition 5.8 since $\mathbb{R}^n$ is Hausdorff (Proposition 4.5). $\square$

5.17 Theorem (Heine-Borel Theorem). For $a \leqslant b$, the subset $[a, b]$ of $\mathbb{R}$ with the usual topology is compact.

Proof. Suppose for contradiction that $\mathcal{F}$ is an open cover for $[a, b]$ with no finite subcover.
Write $I_0 = [a, b]$ and divide this interval into two subintervals $[a, (a + b)/2]$ and $[(a+b)/2, b]$. Then $\mathcal{F}$ is an open cover for each of these subintervals. If $\mathcal{F}$ has a finite subcover for each of the subintervals then their union would be a finite subcover for I_0. So, there is no finite subcover for at least one of the subintervals; let $I_1 = [a_1, b_1]$ be such a subinterval. Notice that $b_1 - a_1 = (b-a)/2$. Repeating this process we get a sequence of subintervals $I_n = [a_n, b_n]$ with $b_n - a_n = (b - a)/2^n$ such that

$$a \leqslant a_1 \leqslant \cdots \leqslant a_n \leqslant a_{n+1} \leqslant \cdots \leqslant b_{n+1} \leqslant b_n \leqslant \cdots \leqslant b_1 \leqslant b$$

and such that $\mathcal{F}$ does not have a finite subcover for each I_n.
Then $(a_n)_{n \geqslant 1}$ is an increasing sequence bounded above by b and so, from the theory of sequences in $\mathbb{R}$, is convergent. Similarly $(b_n)_{n \geqslant 1}$ is a decreasing sequence bounded below by a and so is convergent. However, $b_n - a_n = (b - a)/2^n \to 0$ as $n \to \infty$ and so $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$; let the common limit be α.
Since $\alpha \in I_0$ it must lie in some open set U of the cover $\mathcal{F}$. Since U is an open set in the usual topology on $\mathbb{R}$ it is a neighbourhood of α and so there is some $\varepsilon > 0$ such that $B_\varepsilon(\alpha) = (\alpha-\varepsilon, \alpha+\varepsilon) \subset U$ (by Definition 2.4). Now, choose $n \in \mathbb{N}$ such that $b_n - a_n = (b - a)/2^n < \varepsilon$. Then, since $a_n \leqslant \alpha \leqslant b_n$, $\alpha - a_n < \varepsilon$ and $b_n - \alpha < \varepsilon$ so that $I_n = [a_n, b_n] \subset (\alpha - \varepsilon, \alpha + \varepsilon) \subset U$. This shows that the singleton $\{ U \} \subset \mathcal{F}$ is a finite subcover of $\mathcal{F}$ for I_n contradicting the choice of I_n as an interval for which $\mathcal{F}$ does not have a subcover.
This contradiction shows that every open cover for the interval $I_0 = [a, b]$ does have a finite subcover and so $[a, b]$ is compact. $\square$

5.18 Corollary. For $a_i \leqslant b_i$ for $1 \leqslant i \leqslant n$, the subset $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ with the usual topology is compact.

Proof. This follows from the theorem using Theorem 5.11 and induction. $\square$

5.19 Corollary. A closed bounded subset of $\mathbb{R}^n$ with the usual topology is compact.

Proof. Suppose that $X \subset \mathbb{R}^n$ is closed and bounded. Since X is bounded there is a real number M such that $|\mathbf{x}| \leqslant M$ for all $\mathbf{x} \in X$. Then $|x_i| \leqslant M$ for $1 \leqslant i \leqslant n$ and so $X \subset [-M, M] \times \cdots \times [-M, M] = [-M, M]^n$. $[-M, M]^n$ is compact by Corollary 5.18. Hence, X is compact by Proposition 5.9. $\square$