# 6 The Fundamental Group

**6.1 Definition.** Suppose that $X$ is a topological space and $x_0$, $x_1 \in X$. Write $I = [0,1]$ with the usual topology. Two paths $\sigma_0$ and $\sigma_1 \colon I \to X$ from $x_0$ to $x_1$ are said to be *homotopic*, written $\sigma_0 \sim \sigma_1$, when there exists a continuous map $H \colon I^2 \to X$ such that

$$\begin{aligned} H(s,0) &= \sigma_0(s), \\ H(s,1) &= \sigma_1(s), \\ H(0,t) &= x_0, \\ H(1,t) &= x_1 \end{aligned}$$

for $s$, $t \in I$. We say that $H$ is a *homotopy* between $\sigma_0$ and $\sigma_1$ and indicate this by writing $H \colon \sigma_0 \sim \sigma_1$.

**6.2 Remark.** In the above situation, if we define $\sigma_t \colon I \to X$, for $t \in I$, by $\sigma_t(s) = H(s,t)$, then $\sigma_t$ is a path from $x_0$ to $x_1$ in $X$ and $\{\, \sigma_t \mid t \in I \,\}$ provides a 'continuous family' of such maps betweem $\sigma_0$ and $\sigma_1$.

**6.3 Example.** (a) Given $\mathbf{x}_0$, $\mathbf{x}_1 \in D^2$, all paths in $D^2$ from $\mathbf{x}_0$ to $\mathbf{x}_1$ are homotopic.

*Proof.* Define $H \colon I^2 \to \mathbb{R}^2$ by

$$H(s,t) = (1-t)\sigma_0(s) + t\sigma_1(s).$$

This is a continuous map since $\sigma_0$ and $\sigma_1$ are continuous and $H(I^2) \subset D^2$ since, for $s$, $t \in I$,

$$|H(s,t)| = |(1-t)\sigma_0(s)+t\sigma_1(s)| \leqslant (1-t)|\sigma_0(s)|+t|\sigma_1(s)| \leqslant (1-t)+t = 1.$$

Hence, $H \colon I^2 \to D^2$ and $H \colon \sigma_0 \sim \sigma_1$. $\square$

(b) The same argument works for paths in any convex subset of $\mathbb{R}^n$.

(c) In $S^1 \subset \mathbb{C}$ with the usual topology, the paths $\sigma_0(s) = \exp(i\pi s)$ and $\sigma_1(s) = \exp(-i\pi s)$ are both from 1 to $-1$ but are not homotopic.

The proof of this will be given in Section 7.

**6.4 Proposition.** Homotopy of paths between two points $x_0$ and $x_1$ in a topological space $X$ is an equivalence relation.

*Proof.* We check the conditions for an equivalence relation (Definition 0.15).

*The reflexive property.* For any path $\sigma\colon I \to X$ from $x_0$ to $x_1$ then a homotopy $H\colon \sigma \sim \sigma$ is given by $H(s,t) = \sigma(s)$ (the constant homotopy).

*The symmetric property.* Given a homotopy $H\colon \sigma_0 \sim \sigma_1$ between two paths in $X$ from $x_0$ to $x_1$ then a homotopy $\overline{H}\colon \sigma_1 \sim \sigma_0$ is given by $\overline{H}(s,t) = H(s, 1-t)$ (the reverse homotopy).

*The transitive property.* Given homotopies $H\colon \sigma_0 \sim \sigma_1$ and $K\colon \sigma_1 \sim \sigma_2$ where the $\sigma_i$ are paths in $X$ from $x_0$ to $x_1$ then a homotopy $L\colon \sigma_0 \sim \sigma_2$ is given by

$$L(s,t) = \begin{cases} H(s, 2t) & \text{for } s \in I \text{ and } 0 \leqslant t \leqslant 1/2, \\ K(s, 2t-1) & \text{for } s \in I \text{ and } 1/2 \leqslant t \leqslant 1. \end{cases}$$

This is well defined since, for $t = 1/2$, $H(s,1) = \sigma_1(s) = K(s,0)$. In addition, $L$ is continuous by the Gluing Lemma since $I \times [0,1/2]$ and $I \times [1/2,1]$ are closed subsets of $I^2$. $\square$

**6.5 Definition.** We write $[\sigma]$ for the *homotopy class* of a path $\sigma$ in a topological space $X$. Thus, given two paths $\sigma_0$ and $\sigma_1$ from $x_0$ to $x_1$ in a topological space $X$, $\sigma_0 \sim \sigma_1 \Leftrightarrow [\sigma_0] = [\sigma_1]$.

## The algebra of homotopy classes of paths

**6.6 Proposition.** Given two homotopic paths $\sigma_0 \sim \sigma_1$ from $x_0$ to $x_1$ and two homotopic paths $\tau_o \sim \tau_1$ from $x_1$ to $x_2$ in a topological space $X$, then

$$\sigma_0 * \tau_0 \sim \sigma_1 * \tau_1.$$

*Proof.* Suppose that $H\colon \sigma_0 \sim \sigma_1$ and $K\colon \tau_0 \sim \tau_1$. Then we may define a homotopy $L\colon \sigma_0 * \tau_0 \sim \sigma_1 * \tau_1$ by

$$L(s,t) = \begin{cases} H(2s, t) & \text{for } 0 \leqslant s \leqslant 1/2 \text{ and } t \in I, \\ K(2s-1, t) & \text{for } 1/2 \leqslant s \leqslant 1 \text{ and } t \in I. \end{cases}$$

This is well defined since, for $s = 1/2$, $H(1,t) = x_1 = K(0,t)$. In addition, $L$ is continuous by the Gluing Lemma since $[0,1/2] \times I$ and $[1/2,1] \times I$ are closed subsets of $I^2$. $\square$

**6.7 Definition.** Given a homotopy class $[\sigma]$ of paths from $x_0$ to $x_1$ and $[\tau]$ a homotopy class of paths from $x_1$ to $x_2$ in a topological space $X$ then we define their *product* $[\sigma][\tau]$, a homotopy class of paths from $x_0$ to $x_2$, by

$$[\sigma][\tau] = [\sigma * \tau].$$

This is well-defined by Proposition 6.6.

**6.8 Proposition.** Given a path $\sigma$ from $x_0$ to $x_1$ in a topological space $X$, then

$$\varepsilon_{x_0} * \sigma \sim \sigma \sim \sigma * \varepsilon_{x_1}$$

or, equivalently,

$$[\varepsilon_{x_0}][\sigma] = [\sigma] = [\sigma][\varepsilon_{x_1}].$$

*Proof.* A homotopy $H\colon \varepsilon_{x_0} * \sigma \sim \sigma$ is given by

$$H(s,t) = \begin{cases} x_0 & \text{for } 0 \leqslant s \leqslant (1-t)/2, \\ \sigma\Big(\frac{s-(1-t)/2}{1-(1-t)/2}\Big) & \text{for } (1-t)/2 \leqslant s \leqslant 1. \end{cases}$$

This is well-defined since, for $s = (1-t)/2$, $x_0 = \sigma(0)$. It is continuous by the Gluing Lemma since $\{(s,t) \in I^2 \mid 0 \leqslant s \leqslant (1-t)/2\}$ and $\{(s,t) \in I^2 \mid (1-t)/2 \leqslant s \leqslant 1\}$ are closed subsets of $I^2$.

There is a similar homotopy $\sigma \sim \sigma * \varepsilon_{x_1}$ (Exercise). $\square$

**6.9 Proposition.** Given a path $\sigma$ from $x_0$ to $x_1$ in a topological space $X$, then

$$\sigma * \overline{\sigma} \sim \varepsilon_{x_0} \quad \text{and} \quad \overline{\sigma} * \sigma \sim \varepsilon_{x_1}$$

or, equivalently,

$$[\sigma][\overline{\sigma}] = [\varepsilon_{x_0}] \quad \text{and} \quad [\overline{\sigma}][\sigma] = [\varepsilon_{x_1}].$$

*Proof.* A homotopy $H\colon \sigma * \overline{\sigma} \sim \varepsilon_{x_0}$ is given by

$$H(s,t) = \begin{cases} \sigma\big(2(1-t)s\big) & \text{for } 0 \leqslant s \leqslant 1/2, \\ \sigma\big(2(1-t)(1-s)\big) & \text{for } 1/2 \leqslant s \leqslant 1. \end{cases}$$

This is well-defined since, for $s = 1/2$, both formulae give $\sigma(1-t)$. It is continuous by the Gluing Lemma since $[0,1/2] \times I$ and $[1/2,1] \times I$ are closed in $I^2$.

In a similar way we may write down a homotopy $\overline{\sigma} * \sigma \sim \varepsilon_{x_1}$ [Exercise]. Alternatively, we may apply the first part of the result to the path $\overline{\sigma}$ since $\sigma$ is the reverse of the path $\overline{\sigma}$. $\square$

**6.10 Proposition.** Given two homotopic paths $\sigma_0 \sim \sigma_1$ from $x_0$ to $x_1$ in a topological space $X$, then $\overline{\sigma}_0 \sim \overline{\sigma}_1$.

*Proof.* Exercise. $\square$

**6.11 Definition.** Given a homotopy class $[\sigma]$ of paths from $x_0$ to $x_1$, we define its *inverse* $[\sigma]^{-1}$, a homotopy class of paths from $x_1$ to $x_0$, by $[\sigma]^{-1} = [\overline{\sigma}]$. This is well-defined by Proposition 6.10.

**6.12 Remark.** With this notation we may write the result of Proposition 6.9 as

$$[\sigma][\sigma]^{-1} = [\varepsilon_{x_0}] \quad \text{ and } [\sigma]^{-1}[\sigma] = [\varepsilon_{x_1}].$$

**6.13 Proposition.** Given paths $\sigma$ from $x_0$ to $x_1$, $\tau$ from $x_1$ to $x_2$ and $\rho$ from $x_2$ to $x_3$ in a topological space $X$ then

$$(\sigma * \tau) * \rho \sim \sigma * (\tau * \rho)$$

or, equivalently,

$$([\sigma][\tau])[\rho] = [\sigma]([\tau][\rho]) \quad \text{so that we may write } [\sigma][\tau][\rho] \text{ without ambiguity.}$$

*Proof.* A homotopy $H\colon (\sigma * \tau) * \rho \sim \sigma * (\tau * \rho)$ is given by

$$H(s,t) = \begin{cases} \sigma\big(4s/(1+t)\big) & \text{for } 0 \leqslant s \leqslant (1+t)/4, \\ \tau\big(4(s-(1+t)/4)\big) & \text{for } (1+t)/4 \leqslant s \leqslant (2+t)/4, \\ \rho\Big(\frac{s-(2+t)/4}{1-(2+t)/4}\Big) & \text{for } (2+t)/4 \leqslant s \leqslant 1. \end{cases}$$

This is well-defined since, when $s = (1+t)/4$, $\sigma(1) = x_1 = \tau(0)$ and when $s = (2+t)/4$, $\tau(1) = x_2 = \rho(0)$. It is continuous by the Gluing Lemma since $\{(s,t) \in I^2 \mid 0 \leqslant s \leqslant (1+t)/4\}$, $\{(s,t) \in I^2 \mid (1+t)/4 \leqslant s \leqslant (2+t)/4\}$ and $\{(s,t) \in I^2 \mid (2+t)/4 \leqslant s \leqslant 1\}$ are closed in $I^2$. $\square$

## The algebra of homotopy classes of loops

**6.14 Definition.** Let $X$ be a topological space and $x_0 \in X$. The a *loop* or *closed path* in $X$ *based at* $x_0$ is a path $\sigma\colon I \to X$ from $x_0$ to $x_0$, i.e. such that $\sigma(0) = \sigma(1) = x_0$.

**6.15 Theorem.** The set of homotopy classes of loops in a topological space $X$ based at a point $x_0 \in X$ forms a group under the product $[\sigma][\tau] = [\sigma * \tau]$. The identity is given by $e = [\varepsilon_{x_0}]$ and the inverse $[\sigma]^{-1} = [\overline{\sigma}]$.

*Proof.* This follows from the results of Proposition 6.13 (associative product), Proposition 6.8 ($e$ is an identity element) and Proposition 6.9 ($[\sigma]^{-1}$ is the inverse of $[\sigma]$). $\square$

**6.16 Definition.** This group is called the *fundamental group* of $X$ *with base point* $x_0$ and is denoted $\pi_1(X, x_0)$. [Other names are the *first homotopy group* or the *Poincaré group.* Sometimes the notation $\pi(X, x_0)$ is used.]

**6.17 Remark.** The fundamental group is not necessarily an abelian group.

**6.18 Example.** If $X$ is a convex subset of $\mathbb{R}^n$ with the usual topology and $x_0 \in X$, then $\pi_1(X, x_0) \cong \{e\} = I$, the trivial group.

*Proof.* For all loops $\sigma$ in $X$ based at $x_0$, $\sigma \sim \varepsilon_{x_0}$ by the argument of Example 6.3(b). Hence $[\sigma] = e$. $\square$

## Dependence on the base point

**6.19 Theorem.** Let $X$ be a topological space and $\rho$ be a path in $X$ from $x_0$ to $x_1$. Then $\rho$ induces an isomorphism

$$u_\rho \colon \pi_1(X, x_0) \to \pi_1(X, x_1)$$

by

$$u_\rho(\alpha) = [\rho]^{-1}\alpha[\rho] \quad \text{for } \alpha \in \pi_1(X, x_0).$$

*Proof.* To see that $u_\rho$ is a homomorphism observe that, for $\alpha$, $\beta \in \pi_1(X, x_0)$,

$$\begin{aligned}
u_\rho(\alpha)u_\rho(\beta) &= [\rho]^{-1}\alpha[\rho][\rho]^{-1}\beta[\rho] \\
&= [\rho]^{-1}\alpha[\varepsilon_{x_0}]\beta[\rho] \quad \text{since } [\rho][\rho]^{-1} = [\varepsilon_{x_0}] \text{ (by Remark 6.12)} \\
&= [\rho]^{-1}\alpha\beta[\rho] \quad \text{since } \alpha[\varepsilon_{x_0}] = \alpha \text{ (by Proposition 6.8)} \\
&= u_\rho(\alpha\beta).
\end{aligned}$$

To see that $u_\rho$ is an isomorphism observe that $u_{\overline{\rho}} \colon \pi_1(X, x_1) \to \pi_1(X, x_0)$ provides an inverse since, for $\alpha \in \pi_1(X, x_0)$,

$$\begin{aligned}
u_{\overline{\rho}}u_\rho(\alpha) &= [\overline{\rho}]^{-1}[\rho]^{-1}\alpha[\rho][\overline{\rho}] \\
&= [\rho][\rho]^{-1}\alpha[\rho][\rho]^{-1} \quad \text{since } [\rho]^{-1} = [\overline{\rho}] \text{ and } [\overline{\rho}]^{-1} = [\rho] \text{ (by Definition 2.11)} \\
&= [\varepsilon_{x_0}]\alpha[\varepsilon_{x_0}] \quad \text{since } [\rho][\overline{\rho}] = [\varepsilon_{x_0}] \text{ (by Remark 6.12)} \\
&= \alpha \quad \text{since } [\varepsilon_{x_0}]\alpha = \alpha = \alpha[\varepsilon_{x_0}] \text{ (by Proposition 6.8)}
\end{aligned}$$

so that $u_{\overline{\rho}}u_\rho = I_{\pi_1(X,x_0)} \colon \pi_1(X, x_0) \to \pi_1(X, x_0)$.
And similarly, $u_\rho u_{\overline{\rho}} = I_{\pi_1(X,x_1)} \colon \pi_1(X, x_1) \to \pi_1(X, x_1)$. $\square$

□

**6.20 Remark.** It is clear from the definition that homotopic paths induce the same isomorphism. However, non-homotopic paths may induce different isomorphisms and in that case there is no natural choice of isomorphism. However, this result nevertheless means that, if $X$ is path-connected, then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ for any two base points $x_0$, $x_1 \in X$. In this case we can refer to *the* fundamental group of the space without reference to a base point and this is is sometimes denoted $\pi_1(X)$.

**6.21 Definition.** A topological space $X$ is said to be *simply-connected* when it is path-connected and $\pi_1(X) \cong I$, the trivial group.

## Functorial properties of the fundamental group

**6.22 Theorem.** A continuous map of topological spaces $f\colon X \to Y$ induces a homomorphism
$$f_*\colon \pi_1(X, x_0) \to \pi_1\big(Y, f(x_0)\big)$$
by $f_*([\sigma]) = [f \circ \sigma]$ for any choice of base point $x_0 \in X$. This has the following properties.

(a) The identity map $I_X\colon X \to X$ induces the identity map
$$(I_X)_* = I_{\pi_1(X,x_0)}\colon \pi_1(X, x_0) \to \pi_1(X, x_0).$$

(b) Given continuous maps $f\colon X \to Y$ and $g\colon Y \to Z$ then
$$(g \circ f)_* = g_* \circ f_*\colon \pi_1(X, x_0) \to \pi_1\big(Z, gf(x_0)\big).$$

*Proof.* The function $f_*$ is well-defined since, if $[\sigma_0] = [\sigma_1]$ then $\sigma_0 \sim \sigma_1$ and so there exists a homotopy $H\colon \sigma_0 \sim \sigma_1$. Then $f \circ H\colon I^2 \to Y$ gives a homotopy $f \circ \sigma_0 \sim f \circ \sigma_1$ and so $[f \circ \sigma_0] = [f \circ \sigma_1]$.

To see that $f_*$ is a homomorphism suppose that $[\sigma]$, $[\tau] \in \pi_1(X, x_0)$. Then
$$f_*([\sigma][\tau]) = f_*([\sigma * \tau]) = [f \circ (\sigma * \tau)]$$
and
$$f_*([\sigma])f_*([\tau]) = [f \circ \sigma][f \circ \tau] = [(f \circ \sigma) * (f \circ \tau)]$$

and by writing out the formulae we see that $f\circ(\sigma*\tau) = (f\circ\sigma)*(f\circ\tau)\colon I \to Y$. Hence, $f_*([\sigma][\tau]) = f_*([\sigma])f_*([\tau])$.

(a) Property (a) is immediate from the definition since $I_X \circ \sigma = \sigma$.

(b) Property (b) is immediate from the definition since $(g \circ f) \circ \sigma = g \circ (f \circ \sigma)$. $\square$

**6.23 Corollary.** The fundamental group is a topological invariant: if $f\colon X \to Y$ is a homeomorphism then $f_*\colon \pi_1(X, x_0) \to \pi_1\big(Y, f(x_0)\big)$ is an isomorphism.

*Proof.* This follows immediately from the functorial properties. Suppose that $f\colon X \to Y$ is a homeomorphism with inverse $f^{-1} = g\colon Y \to X$. Then $g_*\colon \pi_1\big(Y, f(x_0)\big) \to \pi_1(X, x_0)$ is the inverse of $f_*$ proving that $f_*$ is an isomorphism. To prove this observe that

$g_*\circ f_* = (g\circ f)_*$ (by 6.22(b)) $= (I_X)_*$ (since $g = f^{-1}$) $= I_{\pi_1(X,x_0)}$ (by 6.22(a))

and similarly $f_* \circ g_* = I_{\pi_1\left(Y,f(x_0)\right)}\colon \pi_1\big(Y, f(x_0)\big) \to \pi_1\big(Y, f(x_0)\big)$. $\square$

**6.24 Remark.** Later we will proof that the fundamental group of the circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ is isomorphic to $\mathbb{Z}$. The intuitive idea is to count how often a given loop winds around the circle. Here, the sign accounts for the orientation of the loop (counter clockkwise vs. clockwise).

**6.25 Remark.** The fundamental group is the *first* homotopy group since the definition can be generalized to a definition of an $n$th homotopy group $\pi_n(X, x_0)$ for each natural number $n$. Whereas the first homotopy group is defined using certain continuous maps $I \to X$ the $n$th homotopy group is defined using certain continuous maps $I^n \to X$.

## Exercises

**1.** (a) Given a path $\sigma\colon I \to X$ from $x_0$ to $x_1$ in a topological space $X$, prove that

$$\sigma * \varepsilon_{x_1} \sim \sigma.$$

[Proposition 6.8, second part]
(b) Given two homotopic paths $\sigma_0 \sim \sigma_1$ from $x_0$ to $x_1$ in a topological space $X$, prove that $\overline{\sigma}_0 \sim \overline{\sigma}_1$. [Proposition 6.10]

**2.** Suppose that $X$ is a convex subset of $\mathbb{R}^n$ with the usual topology [see Problems 1, Question 7.] Prove that, all paths from $x_0$ to $x_1 \in X$ are homotopic. Deduce that $\pi_1(X) \cong I$, the trivial group.

**3.** Suppose that $X$ is a path-connected space and $x_0$, $x_1 \in X$. Prove that all paths from $x_0$ to $x_1$ are homotopic if and only if $X$ is simply-connected.

**4.** Suppose that $X$ is a path-connected topological space and $x_0$, $x_1 \in X$. Prove that all paths $\rho$ from $x_0$ to $x_1$ induce the same isomorphism $u_\rho \colon \pi_1(X, x_0) \to \pi_1(X, x_1)$ if and only if the fundamental group $\pi_1(X)$ is abelian.

**5.** Recall from the proof of Proposition 1.17 that a continuous function $f \colon X \to Y$ of topological spaces induces a function $f_* \colon \pi_0(X) \to \pi_0(Y)$ by $f_*([x]) = [f(x)]$. Which of the following assertions are true in general? Give a proof or counterexample for each.

(a) If $f$ is surjective then $f_*$ is surjective.

(b) If $f$ is injective then $f_*$ is injective.

(c) If $f$ is bijective then $f_*$ is bijective.