7 The Fundamental Group of the Circle

We will take the circle S^1 to be the unit circle centre the origin the complex plane:

$$S^1 = \{\, z \in \mathbb{C} \mid |z| = 1 \,\}.$$

7.1 Theorem. $\pi_1(S^1, 1) \cong \mathbb{Z}$, the additive group of the integers.

The proof of this result will take up the whole of this section. In order to define the isomorphism we need two key results which we will prove towards the end of the section.

7.2 Theorem (The Path-Lifting Theorem). Let $p\colon \mathbb{R} \to S^1$ be the exponential map given by $p(x) = \exp(2\pi i x) \in S^1 \subset \mathbb{C}$. Suppose that $\sigma\colon I \to S^1$ is a loop in S^1 based at 1. Then there is a unique path $\widetilde{\sigma}\colon I \to \mathbb{R}$ such that

(a) $p \circ \widetilde{\sigma} = \sigma$ (we say that $\widetilde{\sigma}$ is a *lift* of σ to $\mathbb{R}$); and

(b) $\widetilde{\sigma}(0) = 0$.

7.3 Definition. Given a loop σ in S^1 based at 1 with lift $\widetilde{\sigma}\colon I \to \mathbb{R}$ such that $\widetilde{\sigma}(0) = 0$. Since $p\widetilde{\sigma}(1) = \sigma(1) = 1$ it follows that $\widetilde{\sigma}(1) \in p^{-1}(1) = \mathbb{Z}$. We define the *degree* of the loop σ to be $\widetilde{\sigma}(1)$, written $\deg(\sigma) = \widetilde{\sigma}(1)$.

7.4 Theorem (The Monodromy Theorem). Suppose that σ_0 and σ_1 are two homotopic loops in S^1 based at 1. Then $\deg(\sigma_0) = \deg(\sigma_1)$.

7.5 Corollary. We may define a function $\phi\colon \pi_1(S^1, 1) \to \mathbb{Z}$ by $\phi([\sigma]) = \deg(\sigma)$.

Proof. The Monodromy Theorem shows that ϕ is well-defined. $\square$

7.6 Proposition. The function $\phi\colon \pi_1(S^1, 1) \to \mathbb{Z}$ is a group isomorphism (hence proving Theorem 7.1).

This will be proved by a sequence of lemmata.

7.7 Lemma. ϕ is a homomorphism.

Proof. Suppose that σ and τ are two loops in S^1 based at 1. Let $\widetilde{\sigma}, \widetilde{\tau}\colon I \to \mathbb{R}$ be lifts of σ, τ respectively such that $\widetilde{\sigma}(0) = \widetilde{\tau}(0) = 0$. Then $\phi([\sigma]) = \deg(\sigma) = \widetilde{\sigma}(1) \in \mathbb{Z}$ and $\phi([\tau]) = \deg(\tau) = \widetilde{\tau}(1) \in \mathbb{Z}$.
$[\sigma][\tau] = [\sigma * \tau]$ and so $\phi([\sigma][\tau]) = \deg(\sigma * \tau)$.
Define $\rho\colon I \to \mathbb{R}$ by

$$\rho(s) = \begin{cases} \widetilde{\sigma}(2s) & \text{for } 0 \leqslant s \leqslant 1/2, \\ \widetilde{\sigma}(1) + \widetilde{\tau}(2s-1) & \text{for } 1/2 \leqslant s \leqslant 1. \end{cases}$$

This is well-defined since, for $s = 1/2$, $\widetilde{\sigma}(1) = \widetilde{\sigma}(1) + \widetilde{\tau}(0)$ since $\widetilde{\tau}(0) = 0$ and is continuous by the Gluing Lemma. The composition $p \circ \rho = \sigma * \tau$ and $\rho(0) = 0$ and so $\rho = \widetilde{\sigma * \tau}$, the unique lift of $\sigma * \tau$ guaranteed by Theorem 7.2. Hence $\phi([\sigma][\tau]) = \deg(\sigma * \tau) = \rho(1) = \widetilde{\sigma}(1) + \widetilde{\tau}(1) = \phi([\sigma]) + \phi([\tau])$ as required. $\square$

7.8 Lemma. ϕ is an epimorphism.

Proof. Suppose that $n \in \mathbb{Z}$. Let the loop $\sigma_n\colon I \to S^1$ be defined by $\sigma_n(s) = \exp(2\pi i n s)$. Then its unique lift $\widetilde{\sigma}_n\colon I \to \mathbb{R}$ with $\widetilde{\sigma}_n(0) = 0$ is given by $\widetilde{\sigma}_n(s) = ns$. Hence $\deg(\sigma_n) = \widetilde{\sigma}_n(1) = n$ and so $\phi([\sigma_n]) = n$. Hence ϕ is a surjection and so, since it is a homomorphism, it is an epimorphism (see Definition 0.27 in the background reading). $\square$

7.9 Lemma. ϕ is a monomorphism.

Proof. The homomorphism ϕ is a monomorphism (i.e. an injection, see Definition 0.27) if the kernel $\ker(\phi) = \{\, \alpha \in \pi_1(S^1, 1) \mid \phi(\alpha) = 0 \,\} = I$ (see Proposition 0.32).
Suppose that $\phi([\sigma]) = 0$. Then $\deg(\sigma) = 0$ and so the unique lift of σ to $\mathbb{R}$ with $\widetilde{\sigma}(0) = 0$ has $\widetilde{\sigma}(1) = 0$ and so is a loop in $\mathbb{R}$ based at 0. Then $\widetilde{\sigma} \sim \varepsilon_0$, the constant loop at 0 by a homotopy $H\colon I^2 \to \mathbb{R}$ defined by $H(s,t) = (1-t)\widetilde{\sigma}(s)$. Then $p \circ H\colon \sigma \sim \varepsilon_1$ and so $[\sigma] = [\varepsilon_1] = e \in \pi_1(S^1, 1)$. Hence $\ker(\phi) = \{e\} = I$, the trivial subgroup and so ϕ is a monomorphism. $\square$

The Lebesgue number

The proofs of Theorems 7.2 and 7.4 make use of a result about coverings of compact subsets of Euclidean spaces.

7.10 Definition. Given a non-empty subset $A \subset \mathbb{R}^n$ and a point $\mathbf{x} \in \mathbb{R}^n$ we define the *distance of* $\mathbf{x}$ *from* A by

$$d(\mathbf{x}, A) = \inf\{\, |\mathbf{x} - \mathbf{a}| \mid \mathbf{a} \in A \,\}.$$

This exists since $|\mathbf{x}-\mathbf{a}| \geqslant 0$ for all $\mathbf{a} \in A$ and so the set $\{\, |\mathbf{x}-\mathbf{a}| \mid \mathbf{a} \in A \,\}$ is bounded below.

7.11 Proposition. The function $\mathbb{R}^n \to \mathbb{R}$ given by $\mathbf{x} \mapsto d(\mathbf{x}, A)$ is a continuous function (for $\mathbb{R}^n$ and $\mathbb{R}$ with the usual topologies).

Proof. Exercise. [Prove that $|d(\mathbf{x}, A) - d(\mathbf{x}', A)| \leqslant |\mathbf{x}-\mathbf{x}'|$ by using the triangle inequality and then use the ε-δ definition of continuity.] $\square$

7.12 Theorem (The Lebesgue Number Lemma). Let $\mathcal{F}$ be an open cover for a compact subspace $X \subset \mathbb{R}^n$ (usual topology). The there is a positive real number $\delta > 0$ (a *Lebesgue number* for the cover) so that for each point $\mathbf{x} \in X$, $B_\delta^X(\mathbf{x}) \subset U$ for some $U \in \mathcal{F}$.

Proof. Let $\mathcal{F}$ be an open cover for the subspace $X \subset \mathbb{R}^n$. Then, if $X \in \mathcal{F}$ we can take δ to be any positive number. So suppose that X is not an element of $\mathcal{F}$. Since X is compact, $\mathcal{F}$ has a finite subcover for X, $\{\, U_1, U_2, \ldots, U_n\}$. For each i, $1 \leqslant i \leqslant n$, put $A_i = X \setminus U_i$ (non-empty by the assumption that $X \notin \mathcal{F}$) and define $f\colon X \to \mathbb{R}$ by

$$f(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} d(\mathbf{x}, A_i).$$

Then f is continuous by Proposition 7.11. Furthermore, $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in X$. To see this, observe that, given $\mathbf{x} \in X$, $\mathbf{x} \in U_i$ for some i and so, since U_i is an open subset of X, there exists $\varepsilon > 0$ so that $B_\varepsilon^X(\mathbf{x}) \subset U_i$ which implies that $d(\mathbf{x}, A_i) \geqslant \varepsilon$ which in turn implies that $f(\mathbf{x}) \geqslant \varepsilon/n > 0$.
However, by the result of Problems 6, Question 8(c), f has a minimum value, say δ, and, since $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in X$, $\delta > 0$ so that $f(\mathbf{x}) \geqslant \delta$ for all $\mathbf{x} \in X$,
Now, for $\mathbf{x} \in X$, $\delta \leqslant f(\mathbf{x}) \leqslant d(\mathbf{x}, A_k)$ where $d(\mathbf{x}, A_k)$ is the largest of the numbers $d(\mathbf{x}, A_i)$ for $1 \leqslant i \leqslant n$. Then $B_\delta^X(\mathbf{x}) \subset U_k$ as required. $\square$

Proofs of the Path-Lifting and Monodromy Theorems

7.13 Lemma. (a) Let $V' = \{\, z \in S^1 \mid z \neq -1 \,\}$ and $U_n' = (n-1/2, n+1/2) \subset \mathbb{R}$. Then $p^{-1}(V') = \bigcup_{n\in\mathbb{Z}} U_n'$ and, for each $n \in \mathbb{Z}$, the restriction $p_n' = p|U_n' \colon U_n' \to V'$ is a homeomorphism.

(b) Let $V'' = \{\, z \in S^1 \mid z \neq 1 \,\}$ and $U_n'' = (n, n+1) \subset \mathbb{R}$. Then $p^{-1}(V'') = \bigcup_{n\in\mathbb{Z}} U_n''$ and. for each $n \in \mathbb{Z}$, the restriction $p_n'' = p|U_n'' \colon U_n'' \to V'$ is a homeomorphism.

Proof. (a) The set theoretic statement is clear since $p^{-1}(-1) = \{\, n + 1/2 \mid n \in \mathbb{Z} \,\}$. The inverse of p'_n is given by $z \mapsto (\log_e z)/2\pi i + n$ where

$$\log_e \colon \mathbb{C} \setminus \{\, z \in \mathbb{R} \mid z \leqslant 0 \,\} \mapsto \{\, x + iy \in \mathbb{C} \mid -\pi < y < \pi \}$$

is the principal logarithm defined on the cut plane. This function is continuous from complex analysis.
(b) This is a similar argument. In this case, $z \mapsto \big(\log_e(-z)\big)/2\pi i + (n+1/2)$ (principal logarithm again) is the inverse of p''_n. $\square$

7.14 Lemma. Given a loop $\sigma \colon I \to S^1$ based at 1, there exits a positive integer n, such that, for each i, $0 \leqslant i \leqslant n-1$, $\sigma([(i/n, (i+1)/n]) \subset V'$ or $\sigma([i/n, (i+1)/n]) \subset V''$.

Proof. Given a loop $\sigma \colon I \to S^1$ in S^1 based at 1, $\{\, \sigma^{-1}(V'), \sigma^{-1}(V'') \,\}$ is an open cover of I and so, since I is compact, there is a Lebesgue number $\delta > 0$ for this open cover. Let n be a positive integer so that $1/n < \delta$. Then, for $0 \leqslant i \leqslant n-1$, $[i/n, (i+1)/n] \subset B^I_\delta(i/n) \subset \sigma^{-1}(V')$ or $\sigma^{-1}(V'')$ by the defining property of the Lebesgue number. Hence $\sigma([i/n, (i+1)/n]) \subset V'$ or V'' as required. $\square$

Proof of the Path-Lifting Theorem. Suppose that $\sigma \colon I \to S^1$ is a loop based at 1. Let n be a positive integer such that for each i, $0 \leqslant i \leqslant n-1$, $\sigma([i/n, (i+1)/n]) \subset V'$ or $\sigma([i/n, (i+1)/n]) \subset V''$ as given by Lemma 7.14. We define $\widetilde{\sigma} \colon I \to \mathbb{R}$ by an inductive process.
For $0 \leqslant i < n$ suppose that a continuous function $\widetilde{\sigma}_i \colon [0, i/n] \to \mathbb{R}$ has been defined so that $p\widetilde{\sigma}_i(s) = \sigma(s)$ for $0 \leqslant s \leqslant i/n$ and $\widetilde{\sigma}(0) = 0$.
This is clearly possible for $i = 0$ since $[0,0] = \{0\}$.
Suppose now that $\sigma([i/n, (i+1)/n]) \subset V^{(k)}$ ($k = 1$ or 2, where $V^{(1)} = V'$ and $V^{(2)} = V''$). Then, using the notation of Lemma 7.13, $\widetilde{\sigma}_i(i/n) \in U^{(k)}_m$ for some $m \in \mathbb{Z}$ so that $\widetilde{\sigma}_i(i/n) = (p^{(k)}_m)^{-1}\sigma(i/n)$. Hence we can define $\widetilde{\sigma}_{i+1} \colon [0, (i+1)/n] \to \mathbb{R}$ by

$$\widetilde{\sigma}_{i+1}(s) = \begin{cases} \widetilde{\sigma}_i(s) & \text{for } 0 \leqslant s \leqslant i/n, \\ \big(p^{(k)}_m\big)^{-1}\sigma(s) \in U^{(k)}_m & \text{for } i/n \leqslant s \leqslant (i+1)/n. \end{cases}$$

This is well-defined and is continuous by the Gluing Lemma and $\widetilde{\sigma}_{i+1} \colon [0, (i+1)/n] \to \mathbb{R}$ is a continuous function such that $p\widetilde{\sigma}_{i+1}(s) = \sigma(s)$ for $0 \leqslant s \leqslant (i+1)/n$ and $\widetilde{\sigma}_{i+1}(0) = 0$.
After n steps,$\widetilde{\sigma} = \widetilde{\sigma}_n \colon [0,1] \to \mathbb{R}$ is the required lift.
For uniqueness, suppose that $\widetilde{\sigma}' \colon I \to \mathbb{R}$ is another lift of σ with $\widetilde{\sigma}'(0) = 0$. Then, for $s \in I$,$p\widetilde{\sigma}(s) = p\widetilde{\sigma}'(s)$ and so $\widetilde{\sigma}(s) - \widetilde{\sigma}'(s) \in \mathbb{Z}$. So we may define a

continuous function $f\colon I \to \mathbb{Z}$ by $f(s) = \widetilde{\sigma}(s) - \widetilde{\sigma}'(s)$. This must be constant by the Intermediate Value Theorem. But $f(0) = 0$ and so $f(s) = 0$ for all $s \in I$, i.e. $\widetilde{\sigma}(s) = \widetilde{\sigma}'(s)$ for all $s \in I$, i.e. $\widetilde{\sigma} = \widetilde{\sigma}'$. □

Proof of the Monodromy Theorem. Suppose that $H\colon \sigma \sim \sigma'$ is a homotopy between two loops in S^1 based at 1. Then we may prove, by a similar argument to that used to prove Theorem 7.2, that there is a lift $\widetilde{H}\colon I^2 \to \mathbb{R}$ of H (i.e. $p \circ \widetilde{H} = H$) with $H(0,0) = 0$ (see below for the details).
Now. for $t \in I$, $p\widetilde{H}(0,t) = H(0,t) = 1$ and so $t \mapsto \widetilde{H}(0,t)$ is a continuous function $I \to \mathbb{Z}$ and so is constant so that $\widetilde{H}(0,1) = \widetilde{H}(0,0) = 0$.
Similarly, $t \mapsto \widetilde{H}(1,t)$ is a continuous function $I \to \mathbb{Z}$ and so is constant so that $\widetilde{H}(1,1) = \widetilde{H}(1,0)$.
But $s \mapsto \widetilde{H}(s,0)$ is a lift of σ_0 with $\widetilde{H}(0,0) = 0$ and so $\deg(\sigma_0) = \widetilde{H}(1,0)$.
Similarly, $s \mapsto \widetilde{H}(s,1)$ is a lift of σ_1 with $\widetilde{H}(0,1) = 0$ and so $\deg(\sigma_1) = \widetilde{H}(1,1)$.
Hence, $\deg(\sigma_0) = \widetilde{H}(1,0) = \widetilde{H}(1,1) = \deg(\sigma_1)$.
[Note that $\widetilde{H}$ gives a homotopy $\widetilde{\sigma}_0 \sim \widetilde{\sigma}_1$.]
To construct the required lift $\widetilde{H}$ of a homotopy H, observe that, as in the proof of Lemma 7.14, $\{\, H^{-1}(V'),\, H^{-1}(V'')\,\}$ is an open cover for I^2, a compact space, and so has a Lebesgue number $\delta > 0$.
Let n be a positive integer so that $1/n < \delta/\sqrt{2}$. Then, for $0 \leqslant i, j \leqslant n-1$,

$$[i/n,(i+1)/n] \times [j/n,(j+1)/n] \subset B^{I^2}_{\delta}(i/n, j/n) \subset H^{-1}(V') \text{ or } H^{-1}(V'')$$

by the defining property of the Lebesgue number. Hence $H([i/n,(i+1)/n] \times [j/n,(j+1)/n]) \subset V'$ or V''.
We can now construct a continuous lift $\widetilde{H}\colon I^1 \to \mathbb{R}$ such that $p \circ \widetilde{H} = H$ and $\widetilde{H}(0,0) = 0$ by a double induction process which can be described informally as follows (the formal details are similar to those in the proof of Theorem 7.2).
First define $\widetilde{H}$ over the small square $[0,1/n] \times [0,1/n]$ so that $p \circ \widetilde{H} = H$ and $H(0,0) = 0$. This can be done since H maps $[0,1/n] \times [0,1/n]$ into V'. Now (using the fact that H maps each small square into either V' or V'') extend $\widetilde{H}$ successively over the squares $[i/n,(i+1)/n] \times [0,1/n]$ for $1 \leqslant i \leqslant n-1$ so that the definitions agree on the common edges of successive squares. This defines $\widetilde{H}$ over the rectangle $[0,1] \times [0,1/n]$. Using the same method we extend $\widetilde{H}$ successively over each rectangle $[0,1] \times [j/n,(j+1)/n]$ for $1 \leqslant j \leqslant n-1$ so that the definitions agree on the common edges of successive rectangle. This gives $\widetilde{H}$ on the whole of $[0,1] \times [0,1]$ as required. □

8 Applications of the Fundamental Group

8.1 Definition. A subspace X_1 of a topological space X is called a *retract* of X if there is a continuous map $r\colon X \to X_1$ such that $r \circ i = I_{X_1}\colon X_1 \to X_1$, the identity map. This is often expressed by saying that there exists a map r so that following diagram commutes (i.e. the functions obtained by composing the functions corresponding to different routes around the diagram from one vertex to another are the same).

$$\begin{tikzcd} X_1 \arrow[rr, "i"] \arrow[dr, "\mathrm{id}_{X_1}"'] & & X \arrow[dl, "r"] \\ & X_1 & \end{tikzcd}$$

The map r is called a *retraction.*

8.2 Example. (a) Given topological spaces X and Y and a point $x_0 \in X$, the subspace $\{x_0\} \times Y$ is a retract of the product space $X \times Y$. A retraction $r\colon X \times Y \to \{x_0\} \times Y$ is given by $r(x, y) = (x_0, y)$ for all $(x, y) \in X \times Y$.

(b) The subspace $\{0, 1\}$ is not a retract of $I = [0, 1]$ with the usual topology since I is path-connected and $\{0, 1\}$ is not (using Theorem 1.9 since a retraction is necessarily a surjection).

8.3 Theorem (The Brouwer Non-Retraction Theorem). The unit circle S^1 is not a retract of the unit disc D^2 (with the usual topology).

Proof. Let $i\colon S^1 \to D^2$ be the inclusion map and suppose for contradiction that there is a retraction $r\colon D^2 \to S^1$. Then $r \circ i = \mathrm{id}_{S^1}\colon S^1 \to S^1$. Hence, by the functorial properties of the fundamental group (Theorem 6.22) we obtain

$$r_* \circ i_* = (r \circ i)_* = (\mathrm{id}_{S^1})_* = \mathrm{id}_{\pi_1(S^1,1)}\colon \pi_1(S^1, 1) \to \pi_1(S^1, 1).$$

But this is impossible since $\pi_1(S^1, 1) \cong \mathbb{Z}$ and $\pi_1(D^2, 1) \cong \{0\}$, the trivial group (written additively to fit in with the other group). More precisely, given non-zero $n \in \mathbb{Z}$, $r_* \circ i_*(n) = r_*(0) = 0 \neq n$. So this is a contradiction and therefore the map r cannot exist. $\square$

8.4 Remark. The above argument using the functorial properties of the fundamental group is often expressed by saying the the first of the following commutative diagrams induces the second.

$$\begin{tikzcd} S^1 \arrow[rr, "i"] \arrow[dr, "\mathrm{id}_{S^1}"'] & & D^2 \arrow[dl, "r"] \\ & S^1 & \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} \pi_1(S^1,1) \arrow[rr, "i_*"] \arrow[dr, "(\mathrm{id}_{S^1})_* = \mathrm{id}"'] & & \pi_1(D^2,1) \arrow[dl, "r_*"] \\ & \pi_1(S^1,1) & \end{tikzcd}$$

8.5 Corollary (The Brouwer Fixed Point Theorem). Given a continuous map $f\colon D^2 \to D^2$ it has a fixed point, i.e. a point $\mathbf{x} \in D^2$ such that $f(\mathbf{x}) = \mathbf{x}$.

Proof. Suppose for contradiction that f is such a map with no fixed points, i.e. $f(\mathbf{x}) \neq \mathbf{x}$ for all $\mathbf{x} \in D^2$. Then we may define a retraction $r\colon D^2 \to S^1$ by defining $r(\mathbf{x})$ to be the point where the continuation of the line segement from $f(\mathbf{x})$ to $\mathbf{x}$ meets S^1.
The map $r\colon D^2 \to S^1$ is continuous. This is intuitively clear but can be proved by writing down an explicit formula for $r(\mathbf{x})$ in terms of $\mathbf{x}$ and $f(\mathbf{x})$ We obtain

$$r(\mathbf{x}) = \left(\frac{A(x_2 - y_2) + (x_1 - y_1)\sqrt{a^2 - A^2}}{a^2}, \frac{-A(x_1 - y_1) + (x_2 - y_2)\sqrt{a^2 - A^2}}{a^2} \right)$$

where $\mathbf{x} = (x_1, x_2)$, $f(\mathbf{x}) = (y_1, y_2)$, $a^2 = |\mathbf{x} - \mathbf{y}|^2$, $A = x_1y_2 - x_2y_1$.
From the definition $r(\mathbf{x}) = \mathbf{x}$ if $\mathbf{x} \in S^1$ and so $r\colon D^2 \to S^1$ is a retraction which cannot exist by Theorem 8.3 and so we have a contradiction. Hence f must have a fixed point. $\square$

8.6 Theorem [The Fundamental Theorem of Algebra]. Any non-constant polynomial equation in the complex numbers $\mathbb{C}$ has a root.

Proof. Suppose for contradiction that $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ where $n > 0$ is a polynomial with complex coefficients ($a_i \in \mathbb{C}$) such that $f(z) \neq 0$ for all $z \in \mathbb{C}$. Write $f(z) = z^n + g(z)$ and more generally define $f_t(z) = z^n + tg(z)$ for $t \in [0,1]$ so that $f_0(z) = z^n$ and $f_1 = f$.
Choose $R > 0$ such that $|z|^n > |g(z)|$ for all z such that $|z| = R$. (This exists because $|z^n/g(z)| \to \infty$ as $|z| \to \infty$.
Define a loop in S^1 based at 1 by

$$\sigma(s) = \frac{f(Re^{2\pi i s})}{|f(Re^{2\pi i s}|)} \frac{|f(R)|}{f(R)}.$$

(Here the division by $|f(Re^{2\pi i s}|$ is to ensure that S^1 is the codomain and the final factor is ensure that the loop is based at 1.)

Define a homotopy $H\colon I^2 \to S^1$ by

$$H(s,t) = \frac{f_t(Re^{2\pi is})}{|f_t(Re^{2\pi is}|)} \frac{|f_t(R)|}{f_t(R)}.$$

Then $H\colon \sigma_n \sim \sigma$ where $\sigma_n(s) = \exp(2\pi nis)$ and so $\deg(\sigma) = \deg(\sigma_n) = n > 0$.
On the other hand let define $K\colon I^2 \to S^1$ by

$$K(s,t) = \frac{f(tRe^{2\pi is})}{|f(tRe^{2\pi is}|)} \frac{|f(tR)|}{f(tR)}.$$

(Notice that this definition used our assumption that $f(z) \neq 0$ for all z in ensuring that the denominators do not vanish.) Then $K\colon \sigma_0 \sim \sigma$ where σ_0 is the constant loop $\sigma_0(s) = 1$ for all $s \in S^1$. Hence $\deg(\sigma) = \deg(\sigma_0) = 0$. These two different values for the degree of the loop give the necessary contradiction. Hence f must have a zero. $\square$

The Hairy Ball Theorem

8.6 Definition. Suppose that $X \subset \mathbb{R}^n$. Then a *vector field* on X is a continuous map $v\colon X \to \mathbb{R}^n$.
If $X = S^{n-1} \subset \mathbb{R}^n$ then $v\colon S^{n-1} \to \mathbb{R}^n$ is a *tangent vector field* if $v(\mathbf{x})$ is perpendicular to $\mathbf{x}$ for all $\mathbf{x} \in S^{n-1}$.

8.7 Example. A non-vanishing tangent vector field on an odd-dimensional sphere $v\colon S^{2n-1} \to \mathbb{R}^{2n}$ is given by $v(x_1, x_2, \ldots, x_{2n-1}, x_{2n}) = (-x_2, x_1, \ldots, -x_{2n}, x_{2n-1})$.

8.8 Theorem (The Hairy Ball Theorem). Suppose that $v\colon S^2 \to \mathbb{R}^3$ is a tangent vector field on S^2. Then $v(\mathbf{x}) = \mathbf{0}$ for some $\mathbf{x} \in S^2$.

Proof. The proof is by contradiction. Suppose for contradiction that $v\colon S^2 \to \mathbb{R}^3$ be a non-vanishing tangent vector field.
Then, by replacing $v(\mathbf{x})$ by $v(\mathbf{x})/|v(\mathbf{x})|$ if necessary we may suppose that $|v(\mathbf{x})| = 1$ for all $\mathbf{x} \in S^1$.
Furthermore, by rotating about the x_1-axis if necessary, we msy suppose that $v(1,0,0) = (0,0,1)$.
Let $H^+ = \{\mathbf{x} \in S^2 \mid x_3 \geqslant 0\}$, $H^- = \{\mathbf{x} \in S^2 \mid x_3 \leqslant 0\}$.
Let $p\colon H^+ \to D^2$ $(\subset \mathbb{R}^2 \cong \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3)$ be stereographic projection from the point $(0,0,-1)$, i.e. $p(x_1,x_2,x_3) = (x_1,x_2)/(1+x_3)$. This is a homeomorphism fixing $S^1 \times \{0\} \cong S^1$. Using it we can find a vector field on D^2 as follows.

For $\mathbf{x} \in H^+$, the α be the angle between $\mathbf{x}$ and $(0,0,1)$. Define $p_{\mathbf{x}} \colon \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal map obtained by rotating through the angle α about the line $(-x_2, x_1, 0)$ so that $p_{\mathbf{x}}(\mathbf{x}) = (0,0,1)$. [This is well-defined apart from when $\mathbf{x} = (0,0,1)$ and in this case $\alpha = 0$ and so $p_{\mathbf{x}} = I_{\mathbb{R}^3}$, the identity map.] Since $p_{\mathbf{x}}(\mathbf{x}) = (0,0,1)$, $p_{\mathbf{x}}$ maps the vectors perpendicular to $\mathbf{x}$ to $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$. So we can define a vector field on $D^2 \subset \mathbb{R}^2$ by $w_1\big(p(\mathbf{x})\big) = p_{\mathbf{x}}\big(v(\mathbf{x})\big) \in \mathbb{R}^2 \times \{0\} \cong \mathbb{R}^2$. Since $v(1,0,0) = (0,0,1)$, $w_1(1,0) = (-1,0)$. Thus

$$w_1 \colon D^2 \to S^1 \text{ such that } w_1(1) = -1 \tag{1}$$

(identifying $\mathbb{R}^2$ with $\mathbb{C}$ as usual).
Similarly, using the stereographic projection $H^- \to D^2$ (from $(0,0,1)$), we obtain a vector field

$$w_2 \colon D^2 \to S^1 \text{ such that } w_2(1) = 1. \tag{2}$$

The equator $S^1 \times \{0\}$ of the sphere S^2 lies in both H^+ and H^- and this enables us to relate the restrictions of w_1 and w_2 to the boundary circle.
Write $f_i = w_i|S^1$ for $i = 1,2$. Then for $\mathbf{z} \in S^1$, $f_2(\mathbf{z})$ is obtained from $f_1(\mathbf{z})$ by reflection in the line $i\mathbf{z}$. Let $\rho_{\mathbf{z}} \colon S^1 \to S^1$ denote reflection in the line $i\mathbf{z}$. Then

$$f_2(\mathbf{z}) = \rho_{\mathbf{z}}\big(f_1(\mathbf{z})\big) \text{ for all } \mathbf{z} \in S^1. \tag{3}$$

We now observe that the three statements (1), (2) and (3) lead to a contradiction.
Using the functorial properties of the fundamental group, the first of the following commutative diagrams implies the second.

$$\begin{tikzcd} S^1 \arrow[rr, "f_1"] \arrow[dr, "i"'] & & S^1 \\ & D^2 \arrow[ur, "w_1"'] & \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} \pi_1(S^1,1) \arrow[rr, "(f_1)_*"] \arrow[dr, "i_*"'] & & \pi_1(S^1,-1) \\ & \pi_1(D^2,1) \arrow[ur, "(w_1)_*"'] & \end{tikzcd}$$

Since $\pi_1(D^2,1)$ is the trivial group, $(f_1)_*$ is the trivial map. In particular, if $\sigma_1 \colon I \to S^1$ is given by $\sigma_1(s) = \exp(2\pi i s)$, then

$$f_1 \circ \sigma_1 \sim \varepsilon_{-1} \colon I \to S^1. \tag{4}$$

Let $H_1 \colon f_1 \circ \sigma_1 \sim \varepsilon_{-1}$. Then, if we define $H_2 \colon I^1 \to S^1$ by $H_2(s,t) = \rho_{\sigma_1(s)}\big(H_1(s,t)\big)$ we find (using equation (3)) that $H_2 \colon f_2 \circ \sigma_1 \sim \sigma_2$ where $\sigma_2(s) = \rho_{\sigma(s)}\big(\varepsilon_{-1}(s)\big) = \exp(4\pi i s)$. But $\deg(\sigma_2) = 2$ and so

$$\deg\big(f_2 \circ \sigma_1\big) = 2. \tag{5}$$

However, just as above for equation (4), we can prove that

$$f_2 \circ \sigma_1 \sim \varepsilon_1 \colon I \to S^1. \tag{6}$$

But this is incompatible with equation (5) since it implies that $\deg(f_2 \circ \sigma_1) = \deg(\varepsilon_1) = 0$. This gives the necessary contradiction showing that a non-vanishing tangent vector field cannot exist. $\square$