Problems 2: Topological Spaces

1. (a) Prove that $(0, 1)$ is open in $\mathbb{R}$.

(b) Prove that $(0, 1]$ is open in $[-1, 1]$ but not in $\mathbb{R}$.

(c) Prove that $\{1\}$ is open in $\mathbb{Z}$, the subset of integers, but not in $\mathbb{R}$.

2. Which of the following collections of subsets, together with the empty set and $\mathbb{R}$ are a topology on $\mathbb{R}$?

(a) the finite subsets of $\mathbb{R}$ (*not* the finite intervals);

(b) the subsets of $\mathbb{R}$ whose complements are finite;

(c) all subsets of the form $(a, \infty) = \{\, x \in \mathbb{R} \mid x > a \,\}$;

(d) all subsets of the form $[a, \infty) = \{\, x \in \mathbb{R} \mid x \geqslant a \,\}$;

(e) all subsets $U \subset \mathbb{R}$ such that $0 \in U$;

(f) all subsets $U \subset \mathbb{R}$ such that $0 \notin U$.

3. Find all topologies on a set of three elements (say $X = \{a, b, c\}$) and divide them into homeomorphism classes.

4. Prove that, if $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous functions of topological spaces, then $g \circ f\colon X \to Z$ is a continuous function. [Proposition 2.13]

5. Prove that a bijection $f\colon X \to Y$ between topological spaces is a homeomorphism if and only if

$$U \text{ is open in } X \Leftrightarrow f(U) \text{ is open in } Y.$$

[Proposition 2.14]

6. Prove that, for a topological space X, the identity function $I_X\colon X \to X$ given by $I_X(x) = x$ for all $x \in X$ is a homeomorphism. [Example 2.17(c)]

7. Prove that, for topological spaces X and Y and a point $a \in Y$, the constant function $c_a \colon X \to Y$ given bt $c_a(x) = a$ for all $x \in X$ is continuous. [Example 2.17(d)]

8. Prove that, if X and Y are topological spaces, then $f \colon X \to Y$ is continuous if and only if

$$A \text{ closed in } Y \Rightarrow f^{-1}(A) \text{ closed in } X.$$

9. Suppose that X and Y are topological spaces and $\mathcal{B}$ is a basis for the topology of Y. Prove that $f \colon X \to Y$ is continuous if and only if

$$V \in \mathcal{B} \Rightarrow f^{-1}(V) \text{ is open in } X.$$

10. Prove that a basis for the usual topology on $\mathbb{R}^n$ is provided by the set of all ε-balls, $\mathcal{B} = \{\, B_\varepsilon(\mathbf{x}) \mid \mathbf{x} \in \mathbb{R}^n \text{ and } \varepsilon > 0 \,\}$.

11. Prove that a collection of subsets $\mathcal{B} \subset \mathcal{P}(X)$ of a set X is a basis for some topology on X if and only if

(a) for each $x \in X$, there is a subset $U \in \mathcal{B}$ such that $x \in U$,

(b) given U_1, $U_2 \in \mathcal{B}$ and $x \in U_1 \cap U_2$, there exists $U \in \mathcal{B}$ such that $x \in U \subset U_1 \cap U_2$.

12. Prove that a topological space X has a proper subset U (i.e. $U \neq \emptyset$ and $U \neq X$) which is both open and closed if and only if there is a continuous surjection $X \to \{0,1\}$ (where $\{0,1\}$ has the usual topology). Prove that such a topological space is not path-connected.

13. Let $X = \{a, b\}$. For which topologies of Examples 2.17(c) is X path-connected?

14. Given a commutative R ring with 1. The set of it's prime ideals (an ideal $\mathfrak{p} \subsetneq R$ is called *prime* if for $fg \in \mathfrak{p}$ it follows that either $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$) is called the spectrum of R and it is denoted by $\mathrm{Spec}(R)$. For an ideal I of R one considers the subset $U_I = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid I \not\subseteq \mathfrak{p}\}$. Show that the subset of the form U_I form a topology for $\mathrm{Spec}(R)$.