MATH31052 *Topology*

Problems 4: Quotient Spaces

1. Prove Proposition 3.15.

2. Let $X = \{\, x \in \mathbb{R}^2 \mid 1 \leqslant |x| \leqslant 2 \,\}$ be an annulus in $\mathbb{R}^2$ with the usual topology. Define a continuous bijection from the quotient space X/S^1 to unit disc D^2 in $\mathbb{R}^2$ with the usual topology.

[It follows from a general result in §5 that such a map is a homeomorphism.]

3. Suppose that $\sim$ is the equivalence relation on I^2 generated by $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, y)$ for all x, $y \in I = [0, 1]$. Prove that there is a continuous bijection $I^2/{\sim} \to S^1 \times S^1$. [Example 3.20(c). It will follow from a general result in §5 that such a map is a homeomorphism.]

4. Define an equivalence relation on the product space $S^1 \times [-1, 1]$ by $(x, t) \sim (x', t')$ if and only if $(\mathbf{x}, t) = (\mathbf{x}', t')$ or $t = t' = 1$ or $t = t' = -1$. Define a continuous bijection from the quotient space $(S^1 \times [-1, 1])/{\sim}$ to S^2 with the usual topology.

[It follows from a general result in §5 that such a map is a homeomorphism.]

5. Define an equivalence relation on $\mathbb{R}^{n+1} \setminus \{0\}$ by $x \sim \lambda x$ for all non-zero real numbers λ. Prove that the quotient space $(\mathbb{R}^{n+1} \setminus \{0\})/{\sim}$ is homeomorphic to P^n, real projective n-space (as defined in 3.21 in the notes).

[This is the classical definition of projective n-space: as the set of lines through the origin in $\mathbb{R}^{n+1}$.]

6. Let $f\colon S^2 \to \mathbb{R}^4$ be defined by

$$f(x_1, x_2, x_3) = (x_1^2 - x_2^2, x_1x_2, x_1x_3, x_2x_3).$$

Prove that f induces a continuous bijection $F\colon P^2 \to F(P^2) \subset \mathbb{R}^4$ where $F(P^2)$ has the usual topology as a subset of $\mathbb{R}^4$.

[It is a little tricky to show that $f(\mathbf{x}) = f(\mathbf{x}') \Rightarrow \mathbf{x} = \pm\mathbf{x}'$ so if this causes problems I suggest that you don't spend too much time on it.

It follows from a general result in §5 that this map is a homeomorphism. Such a homeomorphism is called an *embedding* of the projective plane in $\mathbb{R}^4$. It can be shown that there is no embedding of the projective plane in $\mathbb{R}^3$.]

7. Let $D^n_r(\mathbf{a}) = \{\, \mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x} - \mathbf{a}| \leqslant r \,\}$. Define an equivalence relation on the subset $X = D^2_1(2,0) \cup D^2_1(-2,0)$ of $\mathbb{R}^2$ with the usual topology by $(2,0) + \mathbf{x} \sim (-2,0) + \mathbf{x}$ for $|\mathbf{x}| = 1$. Prove that there is a homeomorphism from the quotient space $X/{\sim}$ to S^2 with the usual topology.

[This shows that gluing two discs together by their boundary circles gives a sphere. You can get a continuous function $X \to S^2$ which induces the homeomorphism by mapping one disc to the upper hemisphere (using the map of $D^2 \to \{\, \mathbf{x} \in S^2 \mid x_2 \geqslant 0 \,\}$ asked for in Problems 1, Question 4) and the other to the lower hemisphere. You can prove that the inverse map is continuous using the Gluing Lemma.]

8. Define an equivalence relation on D^2 by $\mathbf{x} \sim \mathbf{x}' \Leftrightarrow \mathbf{x} = \mathbf{x}'$ or $(\mathbf{x},\ \mathbf{x}' \in S^1$ and $\mathbf{x}' = -\mathbf{x}$. Prove that the there is a continuous bijection from $D^2/{\sim}$ to P^2.

[It follows from a general result in §5 that such a map is a homeomorphism. To construct the bijection you may find it useful to make use of the map $D^2 \to \{\, \mathbf{x} \in S^2 \mid x_2 \geqslant 0 \,\}$ asked for in Problems 1, Question 4.]