Spring Semester 2020–2021

MATH31052 *Topology*

Problems 5: Hausdorff Spaces

1. Suppose that $f\colon X \to Y$ is a continuous function of topological spaces. Prove that, if $a_n \to a$ as $n \to \infty$ in X, then $f(a_n) \to f(a)$ as $n \to \infty$ in Y.

2. Prove that in a Hausdorff space a sequence can have at most one limit (Proposition 4.4).

[Hint: Give a proof by contradiction starting by supposing, for contradiction, that a sequence has two distinct limits.]

3. Prove that a subset $X \subset \mathbb{R}^n$ with the usual topology is Hausdorff (Proposition 4.5).

4. Prove that a set X with the discrete topology is Hausdorff. What about X with the indiscrete topology?

5. Prove that $\mathbb{R}$ with each of the topologies of Problems 2, Question 2(b), Question 2(c), Question 2(e) and Question 2(f) is not Hausdorff.

6. Prove that, if X is a Hausdorff space and $a \in X$, then the singleton subset $\{a\}$ is a closed subset of X (Proposition 4.6).

[Hint: Prove that $X \setminus \{a\}$ is open by writing it as a union of open sets.]

7. Prove that a subspace of a Hausdorff space is Hausdorff (Proposition 4.8(a)).

8. Prove that the disjoint union of two Hausdorff spaces is Hausdorff (Proposition 4.8(b)).

9. Prove that if A is not a closed subset of X then X/A is not a Hausdorff space.

[Hint: Use Proposition 4.6.]

10. Show that a topological space X is Hausdorff if and only if the diagonal $\Delta_X = \{(x, y) \in X \times X \mid x = y\} \subset X \times X$ is closed (with respect to the product topology).