

Problems 6: Compactness

1. Prove that if A is a subset of a topological space X with the indiscrete topology then A is a compact subset.
2. Prove that if K_1 and K_2 are compact subsets of a topological space X then so is $K_1 \cup K_2$. Hence prove, by induction that a finite union of compact subsets of X is compact. Give an example to show that an infinite union of compact subsets need not be compact.
3. Prove that all the subsets of \mathbb{R} are compact in the cofinite topology (the topology of Problems 2, Question 2(b)).
4. In the topology on \mathbb{R} of Problems 2, Question 2(c), prove that
 - (a) a subset of \mathbb{R} is compact if it is compact with respect to the usual topology;
 - (b) the intervals $[a, b]$ (for $a < b$) and $[a, \infty)$ are compact;
 - (c) the intervals $(a, b]$ (for $a < b$) and $(-\infty, b]$ are not compact.
5. Prove that, given closed non-empty subsets A_n , for $n \geq 1$, of a topological space X such that $A_1 \supset A_2 \supset \cdots \supset A_n \supset A_{n+1} \supset \cdots$ and A_1 is compact, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is non-empty.
[Hint: Give a proof by contradiction. Suppose that the intersection is empty and then use the sequence of closed subsets to construct an open cover of A_1 .]
6. Suppose that X is a compact Hausdorff space with a closed subset $A \subset X$ and a point $b \in X$ such that $b \notin A$. Prove that there are disjoint open subsets U and V such that $A \subset U$ and $b \in V$. A space with this property is called a *regular* space.
[Hint: Use the method used in proving Proposition 5.8.]

7. Prove that the continuous bijections constructed in the solutions to Problems 4, Questions 2, 3, 4, 6 and 8 are homeomorphisms.

8. (a) Suppose that $K \subset \mathbb{R}$ is a non-empty compact set in the usual topology (and so closed and bounded). Let $b = \sup K$, the supremum of K . Use the fact that K is closed to prove that $b \in K$. [Recall the definition of the supremum: b is an upper bound for K ($x \leq b$ of all $x \in K$) and is the least upper bound. It exists by the completeness property of the real numbers.]

(b) Prove that if a non-empty compact subset $K \subset \mathbb{R}$ is path-connected then K is a closed interval $[a, b]$ for some real numbers a and b ($a \leq b$).

(c) Prove that if $f: X \rightarrow \mathbb{R}$ is a continuous function on a non-empty path-connected compact space X then $f(X) = [a, b]$ for some real numbers a and b ($a \leq b$).

9. Where does the method of the proof of the Heine-Borel Theorem (Theorem 5.17) fail if the argument is applied to an open covering of an open interval?

[Hint: See what happens if you apply the argument to the open cover $\mathcal{F} = \{(1/n, 1) \mid n \geq 1\}$ of the open interval $(0, 1)$ in \mathbb{R} with the usual topology.]

10. Given a non-compact Hausdorff space (X, τ) consider the set $X^* = X \sqcup \{\infty\}$ and the topology

$$\tau^* = \tau \cup \{X \setminus C \cup \{\infty\} \mid C \subset X \text{ compact}\}.$$

Show that (X^*, τ^*) is a compact topological space (called the one-point compactification).