Spring Semester 2020–2021

MATH31052 *Topology*

Problems 7: The Fundamental Group

1. (a) Given a path $\sigma\colon I \to X$ from x_0 to x_1 in a topological space X, prove that

$$\sigma * \varepsilon_{x_1} \sim \sigma.$$

[Proposition 6.8, second part]
(b) Given two homotopic paths $\sigma_0 \sim \sigma_1$ from x_0 to x_1 in a topological space X, prove that $\overline{\sigma}_0 \sim \overline{\sigma}_1$. [Proposition 6.10]

2. Suppose that X is a convex subset of $\mathbb{R}^n$ with the usual topology [see Problems 1, Question 7.] Prove that, all paths from x_0 to $x_1 \in X$ are homotopic. Deduce that $\pi_1(X) \cong I$, the trivial group.

3. Suppose that X is a path-connected space and x_0, $x_1 \in X$. Prove that all paths from x_0 to x_1 are homotopic if and only if X is simply-connected.

4. Suppose that X is a path-connected topological space and x_0, $x_1 \in X$. Prove that all paths ρ from x_0 to x_1 induce the same isomorphism $u_\rho\colon \pi_1(X, x_0) \to \pi_1(X, x_1)$ if and only if the fundamental group $\pi_1(X)$ is abelian.

5. Recall from the proof of Proposition 1.17 that a continuous function $f\colon X \to Y$ of topological spaces induces a function $f_*\colon \pi_0(X) \to \pi_0(Y)$ by $f_*([x]) = [f(x)]$. Which of the following assertions are true in general? Give a proof or counterexample for each.

(a) If f is surjective then f_* is surjective.

(b) If f is injective then f_* is injective.

(c) If f is bijective then f_* is bijective.