Spring Semester 2020–2021

MATH31052 *Topology*

Problems 8: The Fundamental Group of the Circle and Applications

1. Let $f\colon S^1 \to S^1$ be the map $f(z) = z^k$ where $k \in \mathbb{Z}$. Describe the homomorphism $f_*\colon \pi_1(S^1, 1) \to \pi_1(S^1, 1)$ in terms of the isomorphism $\phi\colon \pi_1(S^1, 1) \to \mathbb{Z}$ (i.e. what is the homomorphism $\phi \circ f_* \circ \phi^{-1}\colon \mathbb{Z} \to \mathbb{Z}$?).

2. Prove that, for a nonempty subset $A \subset \mathbb{R}^n$ with the usual topology, the function $\mathbb{R}^n \to \mathbb{R}$ given by $\mathbf{x} \mapsto d(\mathbf{x}, A)$ is continuous [Proposition 7.11].

3. Which of the following assertions about a continuous map $f\colon X \to Y$ and the induced homomorphism $f_*\colon \pi_1(X, x_0) \to \pi_1\big(Y, f(x_0)\big)$ (where $x_0 \in X$) are true in general? Give a proof or counterexample for each.

(a) If f is surjective then f_* is surjective.

(b) If f is injective then f_* is injective.

(c) If f is bijective then f_* is bijective.

[Hint. The only non-trivial fundamental group obtained in the course is $\pi_1(S^1) \cong \mathbb{Z}$.]

4. Prove that if a subspace $A \subset X$ of a topological space X is a retract of X then, for $a_0 \in A$,

(a) $i_*\colon \pi_1(A, a_0) \to \pi_1(X, a_0)$ is a monomorphism, where $i\colon A \to X$ is the inclusion map; and

(b) $r_*\colon \pi_1(X, a_0) \to \pi_1(A, a_0)$ is an epimorphism, where $r\colon X \to A$ is a retraction.

5. Given topological spaces X and Y with points $x_0 \in X$ and $y_0 \in Y$, let $p_1 \colon X \times Y \to X$ and $p_2 \colon X \times Y \to Y$ be the projection maps. Prove that the function

$$\pi_1\big(X \times Y, (x_0, y_0)\big) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

given by $\alpha \mapsto \big((p_1)_*(\alpha), (p_2)_*(\alpha)\big)$ is an isomorphism.
[Hint: Think about how to write down the inverse. Given loops in X and in Y how do you use them to get a loop in $X \times Y$?]

6. Find the fundamental group of the torus $S^1 \times S^1$ and the cylinder $S^1 \times I$.

7. Prove that if $x \in D^2 \setminus S^1$ then S^1 is a retract of $D^2 \setminus \{x\}$. Hence, prove that the set of points $x \in D^2$ such that $D^2 \setminus \{x\}$ is simply connected is S^1. Deduce that if $f \colon D^2 \to D^2$ is a homeomorphism then $f(S^1) = S^1$.