**Spring Semester 2020–2021**

**MATH31052** *Topology*

# Solutions 1

**1.** We may define a continuous function $f\colon \mathbb{R}^2 \setminus \{\mathbf{0}\} \to Z$ by increasing the length of each vector by 1, i.e. $f(\mathbf{x}) = (|\mathbf{x}| + 1)\mathbf{x}/|\mathbf{x}|$. The inverse map $g\colon Z \to \mathbb{R} \setminus \{\mathbf{0}\}$ is given by decreasing the length of each vector by 1, i.e. $g(\mathbf{y}) = (|\mathbf{y}| - 1)\mathbf{y}/|\mathbf{y}|$. It is easy to check that $g = f^{-1}$ by evaluating $g \circ f$ and $f \circ g$ (try this). Hence $f\colon \mathbb{R} \setminus \{\mathbf{0}\} \to Z$ is a continuous bijection with continuous inverse and so a homeomorphism.

**2.** We may define a continuous function $f\colon S^1 \to T$ by scaling, i.e. $f(\mathbf{x}) = \mathbf{x}/(|x_1| + |x_2|)$ and a function $f\colon T \to S^1$ by scaling $g(\mathbf{y}) = \mathbf{y}/|\mathbf{y}|$. It is easy to check that $g = f^{-1}$ by evaluating $g \circ f$ and $f \circ g$. Hence $f\colon S^1 \to T$ is a continuous bijection with continuous inverse and so a homeomorphism.

**3.** (a) Notice that each point $x \in [a_1, b_1]$ can be written $x = a_1 + (x - a_1)$. We construct a homeomorphism $f\colon [a_1, b_1] \to [a_2, b_2]$ by translating by $a_2 - a_1$ and scaling by $(b_2 - a_2)/(b_1 - a_1)$ so that $f(x) = a_2 + \dfrac{b_2 - a_2}{b_1 - a_1}(x - a_1)$. (Notice that since $x \in [a_1, b_1]$, $0 \leqslant x - a_1 \leqslant b_1 - a_1$ and so $0 \leqslant f(x) - a_2 \leqslant b_2 - a_2$, i.e. $f(x) \in [a_2, b_2]$.) This is clearly continuous and the inverse function $g\colon [a_2, b_2] \to [a_1, b_1]$ is given by the same formula with $a_1$ and $a_2$ interchanged and $b_1$ and $b_2$ interchanged. You can check by explicit calculation that $g$ is inverse to $f$.

(b) A homeomorphism $(0, \pi/2) \to (0, \infty)$ is given by $t \to \tan t$ and a homeomorphism $(0, 1) \to (0, \pi/2)$ is given by scaling $x \to \pi x/2$ (a special case of part (a)). Hence composing these functions gives a homeomorphism $f\colon (0, 1) \to (0, \pi/2) \to (0, \infty)$ given by the formula $f(x) = \tan(\pi x/2)$. The inverse of $\tan\colon (0, \pi/2) \to (0, \infty)$ is given by the principal value of $\tan^{-1}$ and so again composing this with a scaling map gives the inverse of $f$: the function $g\colon (0, \infty) \to (0, \pi/2) \to (0, 1)$ given by $g(y) = 2\tan^{-1}(y)/\pi$. Again you should check by explicit calculation that $g$ is the inverse of $f$.] A homeomorphism $g\colon (0, \infty) \to \mathbb{R}$ is given by $g(x) = \log_e(x)$ with the inverse given by exp.

**4.** The construction of a homeomorphism $f\colon D^n_r(\mathbf{a}) \to D^n_{r'}(\mathbf{a}')$ uses the same method as for Question 3(a). Each point $\mathbf{x} \in D^n_r(a)$ can be written $\mathbf{x} = \mathbf{a} + (\mathbf{x} - \mathbf{a})$. We construct $f$ by translating the centre of the ball by $\mathbf{a}' - \mathbf{a}$ and by scaling the radial vector $\mathbf{x} - \mathbf{a}$ by $r'/r$. This gives the formula $f(\mathbf{x}) = \mathbf{a}' + (r'/r)(\mathbf{x} - \mathbf{a})$ for $f$. This is clearly continuous and if $|\mathbf{x} - \mathbf{a}| < r$ then $|f(\mathbf{x}) - \mathbf{a}'| < r'$ so that, for $\mathbf{x} \in D^n_r(\mathbf{a})$, $f(\mathbf{x}) \in D^n_{r'}(\mathbf{a}')$. The inverse $g\colon D^n_{r'}(\mathbf{a}') \to D^n_r(\mathbf{a})$ is given by the same formula but with $\mathbf{a}$ and $\mathbf{a}'$ interchanged and $r$ and $r'$ interchanged. Again you should check that $g$ is the inverse of $f$.

**5.** We can construct a homeomorphism from the cylinder to the annulus by mapping the circle at height $x_3$ to the circle of radius $1 + x_3$. This leads to the function given by $f(x_1, x_2, x_3) = (1 + x_3)(x_1, x_2)$ which is clearly continuous. (By definition of the cylinder, $(x_1, x_2) \in S^1$, the unit circle centre the origin. Multiplying by $1 + x_3$ gives a point on the circle of radius $1 + x_3$ centre the origin.)

To prove that $f$ is a bijection (which is clear from the construction) we can can down the inverse $g$ given by $g(y_1, y_2) = (y_1/|y|, y_2/|y|, |\mathbf{y}| - 1)$. (Scaling by $1/|\mathbf{y}|$ gives a point on the unit circle. For the inverse you want the circle of radius $|\mathbf{y}|$ in the annulus to go to the circle at height $|\mathbf{y}| - 1$ in the cylinder.) You should check by explicit calculation that $g$ is inverse to $f$.

**6.** A homeomorphism from the semicircle to the closed interval is given by the projection $f(x_1, x_2) = x_1$. The inverse is given by $f^{-1}(x) = (x, \sqrt{1 - x^2})$ (the formula for the second coordinate comes from the fact that the value lies on the circle and has a non-negative second coordinate.
This can then be generalized to a homeomorphism $f\colon \{\, \mathbf{x} \in S^n \mid x_{n+1} \geqslant 0 \,\} \to D^n$ given by
$f(x_1, x_2, \ldots, x_n, x_{n+1}) = (x_1, x_2, \ldots, x_n)$ with inverse $f^{-1}(\mathbf{y}) = (\mathbf{y}, \sqrt{1 - |\mathbf{y}|^2})$.

**7.** The proof is the same as the proof that $D^n$ is path-connected (Proposition 1.7). Given two points $\mathbf{x}$, $\mathbf{x}' \in A$ a path $\gamma\colon [0,1] \to A$ between them is given by $\gamma(t) = (1 - t)\mathbf{x} + t\mathbf{x}'$. This is continuous from basic analysis and it lies in $A$ because $A$ is convex.
The converse is false since, for example, the circle $S^1$ is path-connected but is not convex.

**8.** Suppose that $f\colon X \to Y$ is a continuous surjection and $X$ is path-connected. To deduce that $Y$ is path-connected we must show that for general points $\mathbf{y}$, $\mathbf{y}' \in Y$ there is a path in $Y$ from $\mathbf{y}$ to $\mathbf{y}'$. To do this we first observe that given such points, since $f$ is a surjection, there are points $\mathbf{x}$, $\mathbf{x}' \in X$ such that $f(\mathbf{x}) = \mathbf{y}$ and $f(\mathbf{y}) = \mathbf{y}'$. Now, since $X$ is path-connected, there exists a path in $X$ from $\mathbf{x}$ to $x'$, say $\sigma\colon [0,1] \in X$. But now $f \circ \sigma\colon [0,1] \to Y$ is a path in $Y$ (since the oomposition of continuous functions is continuous) and $(f \circ \sigma)(0) = f(\mathbf{x}) = \mathbf{y}$ and similarly $f \circ \sigma(1) = \mathbf{y}'$ so that $f \circ \sigma\colon [0,1] \to Y$ is a path in $Y$ from $\mathbf{y}$ to $\mathbf{y}'$ as required. Hence $Y$ is path-connected.

**9.** (a) The function $f\colon \mathbb{R} \to S^1$ given by $f(\theta) = (\cos\theta, \sin\theta)$ is a continuous surjection and $\mathbb{R}$ is path-connected (a convex subset of $\mathbb{R}$) and so, by Theorem 1.9, $S^1$ is path-connected.

(b) Spherical polar coordinates give a continuous surjection $f\colon \mathbb{R}^2 \to S^2$ by

$$f(\theta, \phi) = (\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi).$$

The set $\mathbb{R}^2$ is path-connected since it is convex. Hence $S^2$ is path-connected by Theorem 1.9.

[This argument can be generalized for all spheres $S^n$ for $n \geqslant 1$. We can define a continuous surjection $f_n\colon \mathbb{R}^n \to S^n$ by induction on $n$. The standard parametrization of the circle by $(\cos\theta, \sin\theta)$ gives $f_1$. For the inductive step, if $f_k$ is defined then we can parametrize $S^{k+1}$ by $f_{k+1}(\theta_1, \dots, \theta_{k+1}) = (f_k(\theta_1, \dots, \theta_k)\cos\theta_{k+1}, \sin\theta_{k+1})$. Hence, since $\mathbb{R}^n$ is path-connected so is $S^n$.]

**10.** (a) Using the notation in the question and in the proof of Proposition 1.17, we prove that $(g_* \circ f_*) = (g \circ f)_*\colon \pi_0(X) \to \pi_0(Y)$ by evaluating each function as follows. For $[x] \in \pi_0(X)$, $(g_* \circ f_*)([\mathbf{x}]) = g_*\big(f_*([\mathbf{x}])\big) = g_*([f(\mathbf{x})]) = [g\big(f(\mathbf{x})\big)] = [(g \circ f)(\mathbf{x})] = (g \circ f)_*([\mathbf{x}])$.

(b) To evaluate the function $(I_X)_*\colon \pi_0(X) \to \pi_0(X)$ induced by the identity function $I_X\colon X \to X$ we simply use the definition. So for $[x] \in \pi_0(X)$, $(I_X)_*[\mathbf{x}] = [I_X(\mathbf{x})] = [\mathbf{x}]$ which means that $(I_X)_* = I_{\pi_0(X)}\colon \pi_0(X) \to \pi_0(X)$.

[The properties of $\pi_0(X)$ in (a) and (b) are called the *functorial properties*. Many constructions in topology satisfy these properties.]

(c) Now, if $g\colon Y \to X$ is the inverse of a homeomorphism $f\colon X \to Y$, we have $g \circ f = I_X$ and $f \circ g = I_Y$, Hence $g_* \circ f_* = (g \circ f)_*$ (by (a)) $= (I_X)_* = I_{\pi_0(X)}$ (by (b)) and similarly $f_* \circ g_* = I_{\pi_0(Y)}$ so that $g_8$ is the inverse of $f_*$.

**11.** $[0,1]$ has precisely two cut points of type 1 (the two end points), $[0,1)$ only has one, whereas $(0,1)$ doesn't have any. Hence they are are topologically distinct since homeomorphic spaces have the same number of cut points of each type.

**12.** The proof is more or less the same as the proof of Theorem 1.20. Let $f\colon X \to Y$ be a homeomorphism. If $\{p,q\} \subset X$ us an $n$-pair, then $\{f(p), f(q)\} \subset Y$ is also an $n$-pair since $f$ induces a homeomorphism $X \setminus \{p,q\} \to Y \setminus \{f(p), f(q)\}$ which therefore have the same number of path-components by Proposition 1.17. Hence $f$ induces a bijection between the $n$-pairs of $X$ and the $n$-pairs of $Y$ and so the number of $n$-pairs is the same.