Spring Semester 2020–2021

MATH31052 *Topology*

Solutions 2

1. This question is based on Definition 2.4 and refers to the usual topology on $\mathbb{R}$. We determine whether a subset if open by checking whether or not it contains an ε-ball of each of its points.

(a) For each $a \in (0,1)$ let $\varepsilon = \min(a, 1-a) > 0$. Then $B_\varepsilon(a) = (a-\varepsilon, a+\varepsilon) \subset (0,1)$ since $0 \leqslant a-\varepsilon < a+\varepsilon \leqslant 1$. Hence $(0,1)$ is an open subset of $\mathbb{R}$.

(b) Suppose that $a \in (0,1]$. If $0 < a < 1$ then, since $(0,1) \subset (0,1]$, if $\varepsilon = \min(a, 1-a)$, $B_\varepsilon^{[-1,1]}(a) = B_\varepsilon(a) \subset (0,1]$ (as in part (a)). The other point of $(0,1]$ is 1. First of all $B_1^{[-1,1]}(1) = \{\, x \in [-1,1] \mid |x-1| < 1 \,\} = (0,1] \subset (0,1]$. Hence $(0,1]$ is open in $[-1,1]$. On the other hand, for any $\varepsilon > 0$, $B_\varepsilon^{\mathbb{R}}(1) = (1-\varepsilon, 1+\varepsilon) \not\subset (0,1]$ since $1+\varepsilon/2 \in (1-\varepsilon, 1+\varepsilon)$ but $\notin (0,1]$. Hence $(0,1]$ is not open in $\mathbb{R}$.

(c) First of all $B_1^{\mathbb{Z}}(1) = \{1\}$ and so $\{1\}$ is an open subset of $\mathbb{Z}$. However, for any $\varepsilon > 0$, $B_\varepsilon^{\mathbb{R}}(1) = (1-\varepsilon, 1+\varepsilon) \not\subset \{1\}$ and so $\{1\}$ is not an open subset of $\mathbb{R}$.

2. (a) This is not a topology since arbitrary unions of finite sets need not be finite.

(b) This is a topology (the *cofinite topology*). Suppose that $\mathbb{R} \setminus F_1$ and $\mathbb{R} \setminus F_2$ are complements of finite subsets F_i of $\mathbb{R}$. Then $(\mathbb{R} \setminus F_1) \cap (\mathbb{R} \setminus F_2) = \mathbb{R} \setminus (F_1 \cup F_2)$ is the complement of a finite set and so the intersection condition holds. For the union condition, suppose that $\mathbb{R} \setminus F_\lambda$ are complements of finite sets in $\mathbb{R}$ for $\lambda \in \Lambda$. Then $\bigcup_{\lambda \in \Lambda} (\mathbb{R} \setminus F_\lambda) = \mathbb{R} \setminus \bigcap_{\lambda \in \Lambda} F_\lambda$ which is the complement of a finite set since $\bigcap_{\lambda \in \Lambda} F_\lambda \subset F_{\lambda_0}$ for any $\lambda_0 \in \Lambda$.

[Notice that it is a bit easier to do this question using Proposition 2.16. Since the union of two finite subsets is finite and the intersection of any collection of finite subsets is finite, the finite subsets of $\mathbb{R}$ (together with $\mathbb{R}$) are the closed subsets of a topology on $\mathbb{R}$.]

(c) This is a topology. The intersection condition is easy: $(a_1, \infty) \cap (a_2, \infty) = (a, \infty)$ where $a = \max(a_1, a_2)$. The union condition needs a bit of material from real analysis. Suppose that we are given a collection of sets (a_λ, ∞) where $\lambda \in \Lambda$. If the set $\{a_\lambda\}$ is not bounded below then $\bigcup_{\lambda \in \Lambda} (a_\lambda, \infty) = \mathbb{R}$ since for every $x \in \mathbb{R}$ there must be some $a_\lambda < x$. If the set $\{a_\lambda\}$ is bounded below then let a be the greatest lower bound (also known as the imfimum). Then $\bigcup_{\lambda\in\Lambda}(a_\lambda,\infty) = (a,\infty)$. To see this observe that it is clear that the left subset is a subset of the right subset since $a \leqslant a_\lambda$ for all λ. On the other hand, for $x \in (a,\infty)$, $a < x$ and so x is not a lower bound of the a_λ. Hence $x > a_\lambda$ for some λ, i.e. $x \in (a_\lambda,\infty)$ and so $(a,\infty) \subset \bigcup_{\lambda\in\Lambda}(a_\lambda,\infty)$.

(d) This is not a topology since the union condition fails. For example $\bigcup_{a>0}[a,\infty) = (0,\infty)$.

(e) This is a topology (the *included point topology*). For the intersection condition $0 \in U_1$ and $0 \in U_2 \Rightarrow 0 \in U_1 \cap U_2$. For the union condition $0 \in U_\lambda$ for all $\lambda \in \Lambda \Rightarrow 0 \in \bigcup_{\lambda\in\Lambda} U_\lambda$.

(e) This is a topology (the *excluded point topology*). For the intersection condition $0 \notin U_1$ and $0 \notin U_2 \Rightarrow 0 \notin U_1 \cap U_2$. For the union condition $0 \notin U_\lambda$ for all $\lambda \in \Lambda \Rightarrow 0 \notin \bigcup_{\lambda\in\Lambda} U_\lambda$.

3. The subsets of $X = \{a,b,c\}$ are $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a,b\}$, $\{a,c\}$, $\{b,c\}$, X. Each topology will be a collection of these including $\emptyset$ and X. To list these systematically we consider the number of singleton open subsets which must be between 0 and 3.

(a) Three singleton open subsets. The only possibility is

(i) the discrete topology

since by the union condition all three two point subsets must also be in the topology.

(b) Two singleton open subsets, say $\{a\}$ and $\{b\}$. Then by the union condition $\{a,b\}$ must also be an open subset. The possibilities are:

(ii) (three topologies) $\big\{\emptyset, X, \{a\}, \{b\}, \{a,b\}\big\}$ (and two other homeomorphic topologies by choosing other pairs of points),

(iii) (six topologies) $\big\{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}\big\}$ (and five other homeomorphic topologies by permuting a, b and c).

There are no other possibilities since the intersection condition means that $\{a,c\}$ and $\{b,c\}$ cannot both be open subsets since this would imply that $\{c\}$ was open giving case (a).

(c) One singleton open subset, say $\{a\}$. The possibilities are:

(iv) (three topologies) $\big\{\emptyset, X, \{a\}\big\}$ (and two other homeomorphic topologies by choosing the other points),

(v) (six topologies) $\big\{\emptyset, X, \{a\}, \{a,b\}\big\}$ (and five other homeomorphic topologies by permuting a, b and c),

(vi) (three topologies) $\{\emptyset, X, \{a\}, \{a,b\}, \{a,c\}\}$ (and two other homeomorphic topologies by starting from the other points),

(vii) (three topologies) $\{\emptyset, X, \{a\}, \{b,c\}\}$ (and two other homeomorphic topologies by starting from the other points).

There are no other possibilities since the intersection condition means that, for example, $\{a,c\}$ and $\{b,c\}$ cannot both be open subsets.

(d) No singleton open subsets. The possibilities are:

(viii) $\{\emptyset, X\}$, the indiscrete topology,

(ix) (three topologies) $\{\emptyset, X, \{a,b\}\}$ (and two other homeomorphic topologies by selecting the other pairs of points).

There are no other possibilities since the intersection condition means that there cannot be two two point open subsets.

This gives 29 topologies on a set of three elements divided into nine homeomorphism classes.

4. Given continuous functions $f\colon X \to Y$ and $g\colon Y \to Z$, let V be an open subset of Z. Then, since g is continuous $g^{-1}(V)$ is an open subset of Y and so, since f is continuous, $(g \circ f)^{-1}(V) = f^{-1}g^{-1}(V)$ is an open subset of X. Hence $g \circ f\colon X \to Z$ is continuous.

5. Suppose that $f\colon X \to Y$ is a bijection of topological spaces.

(a) For each $V \subset Y$, $V = f(U)$ where $U = f^{-1}(V)$ (since f is a bijection). Hence f is continuous if and only if (V open in $Y \Rightarrow f^{-1}(V)$ open in X) (Definition 2.12) if and only if ($f(U)$ open in $Y \Rightarrow U$ open in X).

(b) Let $g = f^{-1}\colon Y \to X$. Then $g^{-1} = f$. Hence, for $U \subset X$, $g^{-1}(U) = f(U)$. So $g = f^{-1}$ is continuous if and only if (U open in $X \Rightarrow g^{-1}(U)$ open in Y) if and only if (U open in $X \Rightarrow f(U)$ open in Y).

Hence, f us a homeomorphism if and only if (U open in $X \Leftrightarrow f(U)$ open in Y).

6. For a topological space X, $I_X\colon X \to X$ is continuous since for an open subset $V \subset X$, $I_X^{-1}(V) = V$ which is an open subset of X. Hence I_X is a homeomorphism since $I_X^{-1} = I_X$.

7. For topological spaces X and Y and a point $a \in Y$, given an open subset $V \subset Y$, $c_a^{-1}(V) = X$ if $a \in V$ and $c_a^{-1}(V) = \emptyset$ if $a \notin V$ and so in all cases $c_a^{-1}(V)$ is an open subset of X. Hence $c_a\colon X \to Y$ is continuous.

8. Suppose that $f\colon X \to Y$ is continuous. Then, if A is closed in Y, $Y \setminus A$ is open in Y and so $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$ is open in X from which we conclude that $f^{-1}(A)$ is closed in X, as required. The proof of the converse is obtained by interchanging the use of 'open' and 'closed' in the previous sentence.

9. Suppose that $f\colon X \to Y$ is continuous and $\mathcal{B}$ is a basis for the topology of Y. Then, given $V \in \mathcal{B}$, since V is open $f^{-1}(V)$ is open in X. For the converse, suppose that the given condition holds. Then, for an open subset V of Y, since $\mathcal{B}$ is a basis $V = \bigcup V_\lambda$ for certain elements V_λ of $\mathcal{B}$. So $f^{-1}(V) = \bigcup f^{-1}(V_\lambda)$ is a union of open subsets and so is open as required to prove f continuous.

10. Suppose that U is an open subset of $\mathbb{R}^n$ in the usual topology. Then, by Definition 2.4, for each $\mathbf{x} \in U$, there is a real number $\varepsilon_{\mathbf{x}} > 0$ such that $B_{\varepsilon_{\mathbf{x}}}(\mathbf{x}) \subset U$. Then $U = \bigcup_{\mathbf{x} \in U} B_{\varepsilon_{\mathbf{x}}}(\mathbf{x})$ as required. [Example 2.23 is a special case of this.]

11. Suppose that a collection of open subsets $\mathcal{B} \subset \mathcal{P}(X)$ is the basis for a topology τ. Then $\mathcal{B} \subset \tau$ since a basis is a collection of open subsets, and each subset in τ is a union of subsets in $\mathcal{B}$.
(a) Given $x \in X$, since X is open it is a union of subsets in $\mathcal{B}$. Hence, there must be a subset $U \in \mathcal{B}$ such that $x \in U$.
(b) Given $x \in U_1 \cap U_2$ where $U_i \in \mathcal{B}$, since then $U_i \in \tau$, $U_1 \cap U_2 \in \tau$. Hence $U_1 \cap U_2$ is a union of subsets in $\mathcal{B}$ and so there is a subset $U \in \mathcal{B}$ such that $x \in U \subset U_1 \cap U_2$.

For the converse, suppose that $\mathcal{B} \subset \mathcal{P}(X)$ satisfies the condition in the question. Let $\tau \subset \mathcal{P}(X)$ be the collection of subsets of X given by all possible unions of the subsets in $\mathcal{B}$. We need to confirm that τ is a topology on X. (i) $\emptyset \in \tau$ by taking the empty union and $X \in \tau$ by taking the union of all of the subsets in $\mathcal{A}$ (by condition (a)).
(ii) Suppose that V_1, $V_2 \in \tau$. Then for $x \in V_1 \cap V_2$ since V_1 is a union of subsets in $\mathcal{B}$ there is a subset $U_1 \in B$ such that $x \in U_1 \subset V_1$. Similarly, there is a subset $U_2 \in \mathcal{B}$ such that $x \in U_2 \subset V_2$. Hence $x \in U_1 \cap U_2$ and so, by condition (b), there is a subset $U_x \in \mathcal{B}$ such that $x \in U_x \subset U_1 \cap U_2$ and so $x \in U_x \subset V_1 \cap V_2$. It follows that $V_1 \cap V_2 = \bigcup_{x \in V_1 \cap V_2} U_x$ and so $V_1 \cap V_2 \in \tau$ as required by the definition of τ.
(iii) Suppose that $V_\lambda \in \tau$ for $\lambda \in \Lambda$. Then each V_λ is a union of subsets in $\mathcal{B}$ and so $\bigcup_{\lambda \in \Lambda} V_\lambda$ is a union of subsets in $\mathcal{B}$. Hence $\bigcup_{\lambda \in \Lambda} V_\lambda \in \tau$ as required

by the definition of τ
Hence τ is a topology on X.

12. Suppose that $f\colon X \to \{0,1\}$ is a surjection. The usual topology on $\{0,1\}$ is the discrete topology. Hence $\{0\}$ is open in $\{0,1\}$ and so $U = f^{-1}(0)$ is open in X. Similarly $f^{-1}(1)$ is open in X and so $U = f^{-1}(0) = X \setminus f^{-1}(1)$ is closed. U is a proper subset since both $f^{-1}(0)$ and $f^{-1}(1)$ are non-empty because f is a surjection.
For the converse suppose that U is a proper subset of X which is both open and closed. Then we may define $f\colon X \to \{0,1\}$ by $f(x) = 0$ if $x \in U$ and $f(x) = 1$ if $x \notin U$. This is a surjection since U is a proper subset. To see that it is continuous observe that the proper open subsets of $\{0,1\}$ are $\{0\}$ and $\{1\}$. Since $f^{-1}(0) = U$ and $f^{-1}(1) = X \setminus U$ both of which are open in X, f is continuous.

To see that such a space X is not path-connected we use a proof by contradiction. So suppose that X is path-connected. Then let x, x' be points of X such that $f(x) = 0$ and $f(x') = 1$. Since X is path-connected there is a path $\gamma\colon [0,1] \to X$ from x to x'. Composition with the continuous surjection f and the inclusion map $i\colon \{0,1\} \to \mathbb{R}$ gives a function $i \circ f \circ \gamma\colon [0,1] \to X \to \{0,1\} \to \mathbb{R}$ which does not take values between the values 0 and 1, contradicting the Intermediate Value Theorem (Theorem 0.23). [Or alternatively we can say that $f \circ \gamma$ is a continuous surjection from a path-connected space to a non-path-connected space contradicting Theorem 1.9.] Hence X is not path-connected.

[A topological space which does not have a proper open and closed subset is said to be *connected*. So the above argument shows that if a topological space is not connected then it is not path-connected or, equivalently, if it is path-connected then it is connected. It is not difficult to show that a connected open subset of $\mathbb{R}^n$ is path-connected (see Armstrong, Theorem 3.30). However, in general the two notions are different. The standard example of a topological space which is connected but not path-connected is the union of the graph of the function $\sin(1/x)$ and the y-axis (see Armstrong, pages 62 and 63 for the details). Most books (including Armstrong) use connectedness as the basic notion but I prefer to use path-connectedness since it seems a more natural idea.]

13. For X with the discrete topology the map given by $a \mapsto 0$, $b \mapsto 1$ is a continuous surjection (since all maps from a discrete space are continuous) and so, by Question 12, X is not path-connected.

For X with the indiscrete topology define $\gamma\colon [0,1] \to X$ by $\gamma(t) = a$ for $0 \leqslant t < 1$ and $\gamma(1) = b$. This is continuous (since all maps to an indiscrete space are continuous) and gives a path from a to b and so X is path-connected. For X with the Sierpinski topology $\{\emptyset, X, \{a\}\}$, the map γ defined above is continuous since $\gamma^{-1}(\{a\}) = [0,1)$ is open in $[0,1]$ with the usual topology and so X is path-connected.

14. Set $X = \operatorname{Spec}(R)$. We show that the complements

$$V(I) = X \setminus U_I = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid I \subseteq \mathfrak{p}\}$$

fulfil the properties of a collection of closed subsets in Proposition 2.20. Now, $\emptyset = V((1))$ and $X = V((0))$.

One has $V(I) \cup V(J) = V(IJ)$, with $IJ = (ab \mid a \in I, b \in J)$. Note, that $IJ \subset I$ and $IJ \subset J$. Hence, $V(I) \cup V(J) \subset V(IJ)$. For the other inclusion assume that $\mathfrak{p} \in V(IJ)$, i.e. $IJ \subset \mathfrak{p}$. Since $\mathfrak{p}$ is prime either $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$ (else there would be an element $ab \in IJ \subset \mathfrak{p}$ with $a \notin \mathfrak{p}$ and $b \notin \mathfrak{p}$).

For a collection of closed subset $V_\lambda = V(I_\lambda)$ with $\lambda \in \Lambda$. One considers the ideal I generated by all the I_λ, i.e. the smallest ideal which contains all the I_λ. Since I contains all the I_λ it follows $V(I) \subset V(I_\lambda)$. Hence,

$$V(I) \subset \bigcap_\lambda V(I_\lambda).$$

On the other hand, if $\mathfrak{p} \in \bigcap_\lambda V(I_\lambda)$, then it containes all the I_λ but since $\mathfrak{p}$ is an ideal it also has to contain I (by defintion the smallest ideal containing all the I_λ). Hence, $\mathfrak{p} \in V(I)$ and we conclude

$$V(I) = \bigcap_\lambda V(I_\lambda).$$