Spring Semester 2020–2021

MATH31052 *Topology*

Solutions 3

1. Suppose that a subset $U \subset X_2$ is open in the subspace topology on X_2 induced by the subspace topology on X_1. Then $U = V \cap X_2$ where V is open in the subspace topology on X_1. Since V is open in the subspace topology on X_1, $V = W \cap X_1$ where W is open in X. But then $U = V \cap X_2 = (W \cap X_1) \cap X_2 = W \cap X_2$ and so U is open in the subspace topology on X_2 induced by the topology on X.
Conversely, if U is open in the subspace topology on X_2 induced by X, then $U = W \cap X_2$ where W is open in X. But then $U = (W \cap X_1) \cap X_2$. Here $W \cap X_1$ is open in the subspace topology on X_1 and so U is open in the subspace topology on X_2 induced by the subspace topology on X_1.

2. Suppose that $V \subset X$ is an open subset in the subspace topology. Then $V = U \cap X$ where $U \subset \mathbb{R}^n$ is an open subset in $\mathbb{R}^n$ with respect to the usual topology on $\mathbb{R}^n$. To prove that V is open in X in the usual topology let $\mathbf{x}_0 \in V$. Then we must prove that V is a neighbourhood of $\mathbf{x}_0$ in X. Since $V \subset U$, $\mathbf{x}_0 \in U$ and so, since U is an open subest of $\mathbb{R}^n$, U is a neighbourhood of $\mathbf{x}_0$ in $\mathbb{R}^n$ in the usual topology, i.e. there exists $\varepsilon > 0$ so that $B_\varepsilon(\mathbf{x}_0) \subset U$. Hence $B_\varepsilon^X(\mathbf{x}_0) = B_\varepsilon(\mathbf{x}_0) \cap X \subset U \cap X = V$ so that V is a neighbourbood of $\mathbf{x}_0$ in X as required.
Conversely, suppose that $V \subset X$ is an open subset in the usual topology. Then, for each $\mathbf{x} \in V$, there is a real number $\varepsilon_{\mathbf{x}} > 0$ such that $B_{\varepsilon_{\mathbf{x}}}^X(\mathbf{x}) \subset V$. Then $V = \bigcup_{\mathbf{x} \in V} B_{\varepsilon_{\mathbf{x}}}^X(\mathbf{x}) = \bigcup_{\mathbf{x} \in V} \big(B_{\varepsilon_{\mathbf{x}}}(\mathbf{x}) \cap X\big) = \big(\bigcup_{\mathbf{x} \in V} B_{\varepsilon_{\mathbf{x}}}(\mathbf{x})\big) \cap X$. Hence $U = \bigcup_{\mathbf{x} \in V} B_{\varepsilon_{\mathbf{x}}}(\mathbf{x})$ is an open subset of $\mathbb{R}^n$ in the usual topology so that $V = U \cap X$. Hence U is open in X in the subspace topology.

3. Suppose that A_1 is a closed subset of X_1. Then $X_1 \setminus A_1$ is an open subset of X_1. Hence, $X_1 \setminus A_1 = U \cap X_1$ where U is an open subset of X. Then $A_1 = (X \setminus U) \cap X_1 = A \cap X_1$ where A is closed in X as required.
Conversely, if $A_1 = A \cap X_1$ where A is closed in X, then $X_1 \setminus A_1 = (X \setminus A) \cap X_1$ which is therefore open in X_1 and so A_1 is closed in X_1.

4. The argument here is identical to the proofs of Proposition 3.6 and Theorem 3.7 with 'closed' replaced by 'open' throughout.

5. Let $X_1 = [0, \infty)$ and $X_2 = (-\infty, 0)$ with usual topology. Then $X_1 \cup X_2 = \mathbb{R}$. The constant function $c_0 \colon X_1 \to \mathbb{R}$ is continuous and the constant function $c_1 \colon X_2 \to \mathbb{R}$ is continuous (since constant functions are always continuous) but the glued up function is not.

6. The homeomorphism is give by

$$f \colon \mathbb{R} \times S^1 \to \mathbb{R}^2 \setminus \{0\}; \quad (s, x, y) \to e^s \cdot (x, y)$$

with inverse

$$f \colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R} \times S^1; \quad (x, y) \to \left(\ln(|(x, y)|, \frac{(x, y)}{|(x, y)|} \right).$$

8. For $x_0 = (t_0, i)$ and $x_1 = (t_1, j)$ we can use the path $\sigma \colon [0, 1] \to X$ with

$$\sigma(s) = \begin{cases} ((1-s)t_0 + st_1, i) & s \neq 1, \\ ((1-s)t_0 + st_1, j) & s = 1. \end{cases}$$

this is map is continuous, since the composition $p_1 \circ \sigma \colon [0, 1] \to \mathbb{R}$ with $(p_1 \circ \sigma)(s) = (1-s)t_0 + st_1$ is a continuous map and the composition with $p_2 \circ \sigma \colon \{0, 1\}$ is a map to an indiscrete space and therefore continuous as well. Now the universal property of the product topology implies continuity of σ.

9. We may check the conditions for a topology as follows.
(i) $\emptyset \cap X_i = \emptyset$ which is open in X_i (for $i = 1, 2$) and so $\emptyset$ is open in $X_1 \cup X_2$. $(X_1 \cup X_2) \cap X_i = X_i$ which is open in X_i (for $i = 1, 2$) and so $X_1 \cup X_2$ is open in $X_1 \cup X_2$.
(ii) Suppose that U_1 and $U_2 \subset X_1 \cup X_2$ are open in $X_1 \cup X_2$. Then $U_i \cap X_j$ is open in X_j for $j = 1, 2$. Hence $(U_1 \cap U_2) \cap X_j = (U_1 \cap X_j) \cap (U_2 \cap X_j)$ is open in X_j for $j = 1, 2$ and so $U_1 \cap U_2$ is open in $X_1 \cup X_2$.
(iii) Suppose that U_λ is open in $X_1 \cup X_2$ for $\lambda \in \Lambda$. Then $(\bigcup_{\lambda \in \Lambda} U_\lambda) \cap X_j = \bigcup_{\lambda \in \Lambda} (U_\lambda \cap X_j)$ which is open in X_j and so $\bigcup_{\lambda \in \Lambda} U_\lambda$ is open in $X_1 \cup X_2$ as required.
Hence the definition in the question does give a topology on $X_1 \cup X_2$.
The inclusion map $i_j \colon X_j \to X_1 \sqcup X_2$ is continuous since, given an open subset $U \subset X_1 \sqcup X_2$, $i_j^{-1}(U) = U \cap X_j$ which is open in X_j by definition. Hence, if $f \colon X_1 \sqcup X_2 \to Y$ is continuous, the restricted functions $f|X_j = f \circ i_j \colon X_j \to Y$ is continuous since the composition of continuous functions is continuous.

Conversely, suppose that $f\colon X_1 \sqcup X_2 \to Y$ is such that the restrictions $f_j = f|X_j$ are continuous. Then, given an open set $V \subset Y$, the set $f_j^{-1}(V)$ is open in X_j. Hence $f^{-1}(V)$ is open in $X_1 \sqcup X_2$ since $f^{-1}(V) \cap X_j = f_j^{-1}(V)$. Hence f is continuous.