Spring Semester 2020–2021

MATH31052 *Topology*

Solutions 4

1. Suppose that $q\colon X \to Y$ is a surjection from a topological space X to a set Y. To see that Definition 3.14 does define a topology on Y we check the conditions in Definition 2.11.

(i) $q^{-1}(\emptyset) = \emptyset$ and $q^{-1}(Y) = X$ both of which are open in X and so $\emptyset$ and Y are open in Y.

(ii) Suppose that V_1 and V_2 are open subsets of Y. Then $q^{-1}(V_1)$ and $q^{-1}(V_2)$ are open in X and so $q^{-1}(V_1 \cap V_2) = q^{-1}(V_1) \cap q^{-1}(V_2)$ is open in X which means that $V_1 \cap V_2$ is open in Y.

(ii) Suppose that V_λ is an open subsets of Y for $\lambda \in \Lambda$. Then $q^{-1}(V_\lambda)$ is open in X a for all $\lambda \in \Lambda$ and so $q^{-1}(\bigcup_{\lambda\in\Lambda} V_\lambda) = \bigcup_{\lambda\in\Lambda} q^{-1}(V_\lambda)$ is open in X which means that $\bigcup_{\lambda\in\Lambda} V_\lambda$ is open in Y.

(a) This is immediate from teh definition of open subset of Y.

(b) '$\Rightarrow$:' This follows from (a) and Proposition 2.13 (the composition of continuous maps is continuous).

'$\Leftarrow$:' Suppose that $f \circ q\colon X \to Y \to Z$ is continuous. Then, for an open subset $V \subset Z$, $f \circ q)^{-1}(V)$ is open in X. But $(f \circ q)^{-1}(V) = q^{-1}\big(f^{-1}(V)\big)$ and so $f^{-1}(V)$ is open in Y. Hence f is continuous, as required.

2. Define a function $f\colon X \to D^2$ by $f(\mathbf{x}) = (|\mathbf{x}| - 1)\mathbf{x}/|\mathbf{x}|$ for $\mathbf{x} \in X$. Then f is a continuous surjection. Furthermore, $f(\mathbf{x}) = f(\mathbf{x}')$ if and only if $\mathbf{x} = \mathbf{x}'$ or $|\mathbf{x}| = |\mathbf{x}'| = 1$, i.e. $\mathbf{x}$, $\mathbf{x}' \in S^1$. (For if $|f(\mathbf{x})| > 0$ then $f(\mathbf{x}) = \mathbf{y} \Leftrightarrow \mathbf{x} = (|\mathbf{y}| + 1)\mathbf{y}/|\mathbf{y}|$ whereas $f(\mathbf{x}) = 0 \Leftrightarrow |\mathbf{x}| = 1 \Leftrightarrow \mathbf{x} \in S^1$.) So f induces a bijection $F\colon X/S^1 \to D^2$ by $F([\mathbf{x}]) = f(\mathbf{x})$. F is continuous by the universal property of the quotient topology.

3. Define a surjection $f\colon I^2 \to S^1 \times S^1$ by $f(x,y) = \big(\exp(2\pi i x), \exp(2\pi i y)\big)$ where $S^1 = \{\, z \in \mathbb{C} \mid |\mathbf{z}| = 1 \,\}$. This function is continuous by the universal property of the product topology since the component functions are continuous. Hence, by Theorem 3.18, f induces a continuous bijection $F\colon I^2/\!\sim\; \to S^1 \times S^1$ where the equivalence relation $\sim$ is determined by $(x,y) \sim (x',y') \Leftrightarrow f(x,y) = f(x',y')$ which is the equivalence relation in the question since, for t, $t' \in I$, $\exp(2\pi i t) = \exp(2\pi i t') \Leftrightarrow t = t'$ or $t, t' \in \{0,1\}$. [It is possible to prove that the inverse function F^{-1} is continuous using the Gluing Lemma using an argument similar to that given at the end of Example 3.19 but the method to be given in §5 is much simpler.]

4. Define a function $f\colon S^1 \times [-1,1] \to S^2$ by $f(\mathbf{x}, t) = (\sqrt{1-t^2}\mathbf{x}, t)$. Then f is a continuous surjection so that $f(\mathbf{x}, t) = f(\mathbf{x}', t')$ if and only if $(\mathbf{x}, t) \sim (\mathbf{x}', t')$. (For given $\mathbf{y} \in S^2$, $\mathbf{y} = f(\mathbf{x}, t) \Leftrightarrow (y_1, y_2, y_3) = (\sqrt{1-t^2}x_1, \sqrt{1-t^2}x_2, t) \Leftrightarrow t = y_3$ and $\sqrt{1-t^2}(x_1, x_2) = (y_1, y_2) \Leftrightarrow t = y_3$ and (either $t = y_3 = \pm 1$, in which case $(y_1, y_2) = (0, 0)$ and (x_1, x_2) can be any point of S^1, or $t = y_3 \neq \pm 1$, in which case $(x_1, x_2) = (y_1, y_2)/\sqrt{1-y_3^2}$.) So f induces a bjection $F\colon (S^1 \times [-1,1])/\!\sim\, \to S^2$ by $F[(\mathbf{x}, t)] = f(\mathbf{x}, t)$. F is continuous by the universal property of the quotient topology since this tells us that F is continuous if and only if $F \circ q$ is continuous (where $q\colon X \to X/S^1$ is the quotient map) and $F \circ q = f$ which is continuous.

5. Define a continuous function $f\colon \mathbb{R}^{n+1} \setminus \{0\} \to P^n$ by $f(\mathbf{x}) = [\mathbf{x}/|\mathbf{x}|\,]$. Then $f(\mathbf{x}) = f(\mathbf{x}') \Leftrightarrow \mathbf{x}/|\mathbf{x}| = \pm\mathbf{x}'/|\mathbf{x}'| \Leftrightarrow \mathbf{x} = \lambda\mathbf{x}'$ for some non-zero $\lambda \in \mathbb{R} \Leftrightarrow \mathbf{x} \sim \mathbf{x}'$ under the equivalence relation defined in the question. Hence f induces a well-defined injection $F\colon (\mathbb{R}^{n+1} \setminus \{0\})/\!\sim\, \to P^n$ by $F[\mathbf{x}] = f(\mathbf{x})$. F is a surjection since f is a surjection and is continuous by the universal property of the quotient topology. Hence $F\colon (\mathbb{R}^{n+1} \setminus \{0\})/\!\sim\, \to P^n$ is a continuous bijection of topological spaces.
Now we can write down an inverse for F as follows. Define $g\colon S^n \to (\mathbb{R}^{n+1} \setminus \{0\})/\!\sim$ by $g(\mathbf{x}) = [\mathbf{x}]$. Then as usual this induces a well-defined and continuous function $G\colon P^n \to (\mathbb{R}^{n+1} \setminus \{0\})/\!\sim$ and this is an inverse to F so both F and G are homeomorphisms.

6. It is clear from the definition of f that $\mathbf{x} = \pm\mathbf{x}' \Rightarrow f(\mathbf{x}) = f(\mathbf{x}')$. In order to prove that $f(\mathbf{x}) = f(\mathbf{x}') \Rightarrow \mathbf{x} = \pm\mathbf{x}'$ we need to prove that $f(\mathbf{x})$ determines $\mathbf{x}$ up to sign. Now, if none of the coordinates of $\mathbf{x}$ vanish, the second, third and fourth coordinates of $f(\mathbf{x})$ determine the ratios $x_1 : x_2 : x_3$ and so $\mathbf{x} \in S^2$ is determined up to sign. Now suppose that $x_3 = 0$. Then the values of x_1^2 and x_2^2 are determined by $x_1^2 + x_2^2 = 1$ and the first coordinate of $f(\mathbf{x})$. If in addition $x_1 = 0$ or $x_2 = 0$ this determines $\mathbf{x}$ up to sign. Otherwise, the second coordinate of $f(\mathbf{x})$ gives x_1x_2 and so determines $\mathbf{x}$ up to sign.
Hence f induces an injection $F\colon P^2 = S^2/\!\sim\, \to \mathbb{R}^4$ which is continuous by the universal property of the quotient topology. Thus $F\colon P^2 \to F(P^2)$ is a continuous bijection to the subset $f(P^2)$ of $\mathbb{R}^4$ with the usual topology.

7. Define a continuous surjection $f\colon X = D^2_1(2,0) \cup D^2_1(-2,0) \to S^2$ by $f\big((2,0)+\mathbf{x}\big) = (\mathbf{x}, +\sqrt{1-|\mathbf{x}|^2})$ and $f\big((-2,0)+\mathbf{x}\big) = (\mathbf{x}, -\sqrt{1-|\mathbf{x}|^2})$ for $\mathbf{x} \in D^2$. This is a surjection and $f(\mathbf{a}) = f(\mathbf{a}') \Leftrightarrow \mathbf{a} \sim \mathbf{a}'$ as defined in the question. So, as usual, f induces a continuous bijection $F\colon X/\!\sim\, \to S^2$.
To see that F is a homeomorphism we observe that F has a continuous inverse. The inverse of F is given by

$$F^{-1}(y_1, y_2, y_3) = \begin{cases} [(2,0) + (y_1, y_2)] & \text{for } y_3 \geqslant 0, \\ [(-2,0) + (y_1, y_2)] & \text{for } y_3 \leqslant 0. \end{cases}$$

This is well-defined (since the equivalence relation ensures that the two formulae for points with $y_3 = 0$ give the same equivalence class). It is continuous by the Gluing Lemma since it is continuous on the closed subsets $\{\, \mathbf{y} \mid y_3 \geqslant 0 \,\}$ and $\{\, \mathbf{y} \mid y_3 \leqslant 0 \,\}$ of S^2.

8. Let $f\colon D^2 \to S^2$ be the function $f(\mathbf{x}) = (\mathbf{x}, \sqrt{1-|\mathbf{x}|^2}$. Then f gives a homeomorphism between D^2 and the 'upper' hemisphere $\{\, \mathbf{y} \in S^2 \mid y_3 \geqslant 0 \,\}$ (see Problems 1, Question 6). By taking the composition with the quotient map $q\colon S^2 \to P^2$ we get a continuous surjection $q \circ f\colon D^2 \to P^2$ since each equivalence class in P^2 contains (at least) one point in the image of f. Since f is a injection and q maps each pair of antipodal points to the same point, $q \circ f(\mathbf{x}) = q \circ f(\mathbf{x}') \Leftrightarrow \mathbf{x}' = \mathbf{x}$ or $f(\mathbf{x}') = -f(\mathbf{x}) \Leftrightarrow \mathbf{x}' = \mathbf{x}$ or $\mathbf{x}' = -\mathbf{x} \in S^1$ (since the only pairs of antipodal points in the image of f are equatorial points) $\Leftrightarrow \mathbf{x}' \sim \mathbf{x}$ as given in the question. Hence, as usual, by the univeral property of the qotient topology, $q \circ f\colon D^2 \to P^2$ induces a continuous bijection $F\colon D^2/\!\sim\, \to P^2$ by $F([\mathbf{x}]) = q \circ f(\mathbf{x}) = [f(\mathbf{x})]$.

[Spelling out the details of the proof that F is continuous in this case, let $q_1\colon D^2 \to D^2/\!\sim$ be the quotient map. Then, by the universal property, F is continuous if and only if $F \circ q_1$ is continuous and, by the definition of F, $F \circ q_1 = q \circ f$ which is continuous. There are two quotient maps here. We use the continuity of q and we apply the universal property to q_1.]

[We can use this result to prove that the Example 3.20(f) gives a space homeomorphic to the projective plane by showing that it is homeomorphic to the quotient space in this question. Notice that the square $[-1,1] \times [-1,1]$ is given by $\{\, \mathbf{x} \in \mathbb{R}^2 \mid \max(|x_1|, |x_2|) \leqslant 1 \,\}$. So a homeomorphism $D^2 \to [-1,1]$ is given by $\mathbf{x} \mapsto |\mathbf{x}|\mathbf{x}/\max(|x_1|, |x_2|)$ which has inverse $\mathbf{y} \mapsto \max(|y_1|, |y_2|)\mathbf{y}/|\mathbf{y}|$. And a homeomorphism $[-1,1] \times [-1,1] \to [0,1] \times [0,1]$ is given by $(y_1, y_2) \mapsto \big((y_1+1)/2, (y_2+1)/2\big)$. The composition of these two homeomorphisms gives a homeomorphism $g\colon D^2 \to I^2$ such that $\mathbf{x} \sim \mathbf{x}'$

(under the equivalence relation of this question) if and only if $g(\mathbf{x}) \sim g(\mathbf{x}')$ (under the equivalence relation of 3.20(f)).
Writing $q_1\colon D^2 \to D^2/{\sim}$ and $q_2\colon I^2 \to I^2/{\sim}$ for the quotient maps, $q_2 \circ g\colon D^2 \to I^2/{\sim}$ induces a continuous bijection $G\colon D^2/{\sim} \to I^2/{\sim}$ by $G([\mathbf{x}]) = q_2 \circ g(\mathbf{x}) = [g(\mathbf{x})]$ in the usual way. Similarly, $q_1 \circ g^{-1}\colon I^2 \to D^2/{\sim}$ induces continuous bijection $H\colon I^2/{\sim} \to D^2/{\sim}$ by $H([\mathbf{y}]) = q_1 \circ g^{-1}(\mathbf{y}) = [g^{-1}\mathbf{y}]$. By definition H is the inverse of G and so G is a homeomorphism.
This seems very complicated but the basic idea is simple. In Definition 3.20(f) we form a quotient space by identifying opposite points on the boundary of a unit square. In Question 8, we form a quotient space by identifying opposite points on the boundary of a unit disc. It is pretty clear that there is a homeomorphism from a unit square to a unit disc under which opposite boundary points map to opposite boundary points and so the resulting quotient spaces are homeomorphic.]