Solutions 6

1. In the indiscrete topology on X there are only two open sets ($\emptyset$ and X) and so any open cover of any subset A is already finite. Hence A is compact.

2. Suppose that K_1 and K_2 are compact subsets and $\mathcal{F}$ is an open cover for $K_1 \cup K_2$. Then $\mathcal{F}$ is an open cover for K_1 and so has a finite subcover $\mathcal{F}_1$ for K_1 and similarly $\mathcal{F}$ has a finite subcover $\mathcal{F}_2$ for K_2. Then $\mathcal{F}_1 \cup \mathcal{F}_2$ is a finite subcover for $K_1 \cup K_2$. Hence $K_1 \cup K_2$ is compact.

Now suppose that K_i are compact subsets for $1 \leqslant i \leqslant n$. We prove that $\bigcup_{i=1}^n K_i$ is compact by induction on n. The result is trivial for $n = 1$. Suppose as inductive hypothesis that the result is true for $n = k$. The it follows that it true for $n = k+1$ by the above argument since $\bigcup_{i=1}^{k+1} K_i = \left(\bigcup_{i=1}^k K_i\right) \cup K_{k+1}$ proving the inductive step. Hence the result is true for all n.

For a counterexample, the singleton subset $\{n\}$ in $\mathbb{R}$ with the usual topology is compact but the union $\bigcup_{n \in \mathbb{Z}} \{n\} = \mathbb{Z}$ is not compact.

3. Suppose that $\mathcal{F}$ is an open cover for a non-empty open subset $A \subset \mathbb{R}$ in the cofinite topology. For $a \in A$, $a \in U_0$ for some open subset U_0 in $\mathcal{F}$. Then either $U_0 = \mathbb{R}$, in which case $A \subseteq U_0$ so that $\{U_0\}$ is a finite subcover for A, or $U_0 = \mathbb{R} \setminus \{x_1, x_2, \ldots, x_n\}$ the complement of some finite set $\{x_1, x_2, \ldots, x_n\}$. Reordering this finite set if necessary, suppose that $\{x_1, \ldots, x_k\}$ are the points of the finite set which lie in A. For each such point, since $\mathcal{F}$ is an open cover for A there must be some open subset $U_i \in \mathcal{F}$ such that $x_i \in U_i$. Then $\{\, U_i \mid 0 \leqslant i \leqslant k \,\}$ is a finite subcover for A as required to prove that $\mathbb{R}$ with this topology is compact.

4. (a) This is immediate since each open subset in this topology is open in the usual topology.

(b) Suppose that $\mathcal{F}$ is an open cover for $[a, b)$. Then there must be an open set $U \in \mathcal{F}$ such that $a \in U$. But, by the definition of the topology, $U = (a_1, \infty)$ for some $a_1 \in \mathbb{R}$. Since $a \in (a_1, \infty)$ we must have $a_1 < a$ and so $[a, b) \subseteq (a_1, \infty)$ so that $\{U\}$ is a finite subcover for $[a, b)$ consisting of a single open subset. Hence $[a, b)$ is compact. Similarly, $[a, \infty)$ is compact.

(c) The open covering $\{\, (a+1/n, \infty) \mid n \geqslant 1 \,\}$ for $(a, b]$ has no finite subcover

so $(a, b]$ is not compact.
The open covering $\{ (-n, \infty) \mid n \geqslant 1 \}$ for $(-\infty, b]$ has no finite subcover and so $(-\infty, b]$ is not compact.

5. Suppose for contradiction that the subsets A_n are as in the question but that $\bigcap_{n=1}^{\infty} A_n = \emptyset$. Then $\bigcup_{n=1}^{\infty}(X \setminus A_n) = X \setminus \bigcap_{n=1}^{\infty} A_n = X$ and so $\{ X \setminus A_n \mid n \geqslant 1 \}$ is an open cover for X and so for A_1. But now, since A_1 is compact, there must be a finite subcover of A_1. Now notice that the nesting of the subsets A_n means that $X \setminus A_1 \subseteq X \setminus A_2 \subseteq \cdots \subseteq X \setminus A_n \subseteq X \setminus A_{n+1} \subseteq \cdots$. Let $X \setminus A_k$ be the largest subset in the finite subcover. Then $A_1 \subseteq X \setminus A_k$ (since $X \setminus A_n \subseteq X \setminus A_k$ for $n \leqslant k$ by the nesting of the subsets A_n). Hence $A_k \subseteq A_1 \subseteq X \setminus A_k$ which implies that $A_k = A_k \cap A_1 = \emptyset$ contradicting the choice of the sets A_n as non-empty subsets. Hence, $\bigcap_{n=1}^{\infty} A_n$ is non-empty as required.

[This result is important in dynamical systems.]

6. Suppose that A is a closed subset of a compact Hausdorff space X and b is a point of X such that $b \notin A$. Since X is a Hausdorff space, for each point $a \in A$ there are disjoint open subsets U_a and V_a such that $a \in U_a$ and $b \in V_a$. Then the collection of open subsets $\{ U_a \mid a \in A \}$ is an open cover for A since each point of A lies in U_a, one of the open subsets in the covering. Now, since X is compact and A is a closed subset, A is compact (Proposition 4.6). Hence there is a finite subcover $\{ U_{a_i} \mid 1 \leqslant i \leqslant n \}$ for A. This means that $A \subseteq \bigcup_{i=1}^{n} U_{a_i} = U$, say. U is a union of open subsets and so is an open subset. Now let $V = \bigcap_{i=1}^{n} V_{a_i}$. Then V is a finite intersection of open subsets and so is open. By definition $b \in V_a$ for all a and so $b \in V$. Finally, U and V are disjoint. To see this, observe that, for $1 \leqslant i \leqslant n$, $U_{a_i} \cap V_{a_i} = \emptyset$ and so $U_{a_i} \cap V = \emptyset$ since $V \subseteq V_{a_i}$. Hence $U \cap V = \emptyset$ by the definition of U. Hence U and V are disjoint open subsets as required.

[Notice that we needed the compactness of A in order to get a finite intersection. Only a *finite* intersection of open subsets can be guaranteed to be open. This style of proof is one of the most important applications of compactness.]

7. (a) Problems 4, Question 2. X is compact because it is a closed bounded set in $\mathbb{R}^2$ with the usual topology and so $X/S^1 = q(X)$ is compact since is the continuous image of a compact set. D^2 is Hausdorff since it is a subspace of $\mathbb{R}^2$ with the usual topology. Hence the function $F \colon X/S^1 \to D^2$

is a continuous bijection from a compact space to a Hausdorff space and is therefore a homeomorphism.

(b) Problems 4, Question 3. $S^1 \subset \mathbb{R}^2$ and $[-1,1] \subset \mathbb{R}$ are compact as closed bounded sets in Euclidean spaces with the usual topology and so $S^1 \times [-1,1]$ is compact as the product of two compact spaces. Hence the identification space in the question is compact since it is the continuous image of a compact set. S^2 is Hausdorff since it is a subspace of $\mathbb{R}^3$ with the usual topology. Hence the function F in the solution is a continuous bijection from a compact space to a Hausdorff space and is therefore a homeomorphism.

(c) Problems 4, Question 4. S^1 and $[-1,1]$ are compact (both closed bounded sets in Euclidean spaces with the usual topology) and so the product $S^1 \times [-1,1]$ is compact. Hence $S^1 \times [-1,1]/{\sim} = q(S^1 \times [-1,1])$ is compact. S^1 is Hausdorff (subspace of Euclidean space) and so $S^1 \times S^1$ is Hausdorff. Hence the function F in the solution is a continuous bijection from a compact space to a Hausdorff space and is therefore a homeomorphism.

(d) Problems 4, Question 6. P^2 is the continuous image of compact S^2 and so is compact. $F(P^2)$ is a subspace of $\mathbb{R}^4$ and so is Hausdorff. Hence the continuous bijection $F\colon P^2 \to F(P^2)$ is a homeomorphism.

(e) Problems 4, Question 8. $D^2 \subset \mathbb{R}^2$ is compact as a closed bounded subset of a Euclidean space with the usual topology and so the identification space in the question is compact since it is the continuous image of a compact set. P^2 is Hausdorff because, by Problems 4, Question 6 and Problems 7, Question 7(d), it is homeomorphic to a subspace of $\mathbb{R}^4$ with the usual topology. Hence the continuous function F in the solution is a continuous bijection from a compact space to a Hausdorff space is a homormorphism.

8. (a) We prove that $b \in K$ by contradiction. Suppose for contradiction that $b \notin K$ so that $b \in \mathbb{R} \setminus K$, an open set. Then by the definition of the open sets in the usual topology there is a real number $\varepsilon > 0$ such that $(b - \varepsilon, b + \varepsilon) \subseteq \mathbb{R} \setminus K$ so that $b - \varepsilon$ is an upper bound for K, contradicting the definition of b as the *least* upper bound. Hence $b \in K$.

(b) By a similar argument, if $a = \inf K$ (the greatest lower bound) then $a \in K$. Hence $K \subseteq [a, b]$, $a \in K$ and $b \in K$. However, since K is a path-connected there is a path in K from a to b and by the Intermediate Value Theorem (Theorem 0.23) every point of $[a, b]$ lies on this path. Hence $[a, b] \subseteq K$. Hence $K = [a, b]$.

(c) If $f : X \to \mathbb{R}$ is a continuous function from a non-empty path-connected compact space X then $f(X)$ is a non-empty path-connected compact subset of $\mathbb{R}$ since the continuous image of a path-connected space is path connected (Problems 1, Question 5) and the continuous image of a compact set is compact (Proposition 4.3). Hence $f(X)$ is a closed interval $[a, b]$ by part (b).

9. In the example given, $I_0 = (0, 1)$ which divides into two subintervals $(0, 1/2]$ and $[1/2, 1)$. The second of the subintervals lies in $(1/3, 1)$ and so there is a finite subcover for this subinterval. However, there is not for $(0, 1/2]$ and so $a_1 = 0$ and $b_1 = 1/2$. This interval divides into two subintervals $(0, 1/4]$ and $[1/4, 1/2]$. The second of these lies in $(1/5, 1)$ but there is no subcover for the first. Hence $a_2 = 0$ and $b_2 = 1/4$. Continuing in this way we find that $a_n = 0$ for all n and $b_n = 1/2^n$ so that $\alpha = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \alpha = 0$. At this point the proof breaks down because $\alpha = 0 \notin (0, 1)$ and $\alpha = 0$ does not lie in any open set of the cover.

10. First check that τ^* is a topology. Indeed $X^* = X \setminus \emptyset \cup \{\infty\} \in \tau^*$ and $\emptyset \in \tau \subset \tau^*$. Moreover, if $U, V \in \tau^*$ then $U \cap V \in \tau^*$ this is clear if both are in τ. Assume $U \in \tau$ and $V = X \setminus C \cup \{\infty\}$ then

$$U \cap V = U \cap (X \setminus C) \in \tau \subset \tau^*,$$

since C is closed in X by the Hausdorff property. Assume $U = X \setminus C \cup \{\infty\}$ and $V = X \setminus K \cup \{\infty\}$ then

$$U \cap V = X \setminus (C \cup K) \cup \{\infty\} \in \tau^*,$$

since a finite union of compact sets is compact by Problem 6.2.

Now, consider a union of open sets in τ^*.

$$\bigcup_\lambda U_\lambda \cup \bigcup_\mu V_\mu.$$

with $U_\lambda \in \tau$ and

$$V_\mu = X \setminus C_\mu \cup \{\infty\}.$$

Now, $U = \bigcup_\lambda U_\lambda \in \tau \subset \tau^*$ since τ is a topology and

$$V = \bigcup_\mu V_\mu = X \setminus \bigcap_\mu C_\mu \cup \{\infty\} \in \tau^*,$$

since $\bigcap_\mu C_\mu$ is a closed subset of a compact set (of every C_μ), hence, it is compact. Now it remains to show that $U \cup V$ for $U \in \tau$ and $V = X \setminus C \cup \{\infty\}$ is open:

$$U \cup V = X \setminus (C \cap (X \setminus U)) \cup \{\infty\}.$$

But $(X \setminus U)$ is closed in X hence $(C \cap (X \setminus U) \subset C$ is a closed subset of a compact set. Hence, it is compact and $X \setminus (C \cap (X \setminus U)) \cup \{\infty\} \in \tau^*$.

Now, consider an open cover $\mathcal{F}$ of X^*. In order to cover ∞ it has to include at least one open subset U_∞ of the form $X \setminus C \cup \{\infty\}$ where $C \subset X$ is compact. Now, $\mathcal{F}' = \{U \cap X \mid U \in \mathcal{F}\}$ is an open cover of X (since U and X are open in X^*) and hence of C.

By compactness of C a finite subcover $\{U_1 \cap X, \dots, U_m \cap X\} \subset \mathcal{F}$ suffices to cover C. But then one has the finite subcover $\{U_\infty, U_1, \dots, U_m\} \subset \mathcal{F}$.