Spring Semester 2020–2021

MATH31052 *Topology*

Solutions 7

1. (a) We define a homotopy $H\colon I^2 \to X$ by

$$H(s,t) = \begin{cases} \sigma\left(\dfrac{s}{(1+t)/2}\right) & \text{for } 0 \leqslant s \leqslant (1+t)/2, \\ x_1 & \text{for } (1+t)/2 \leqslant s \leqslant 1. \end{cases}$$

This is well-defined and is continuous by the Gluing Lemma. It gives the required homotopy $H\colon \sigma * \varepsilon_{x_1} \sim \sigma$.

(b) Suppose that $H\colon \sigma_0 \sim \sigma_1$. Define $\overline{H}\colon I^2 \to X$ by $\overline{H}(s,t) = H(1-s,t)$. Then $\overline{H}\colon \overline{\sigma}_0 \sim \overline{\sigma}_1$.

2. Given two paths σ_0 and σ_1 from x_0 to x_1 in X we may define a homotopy $H\colon \sigma_0 \sim \sigma_1$ by $H(s,t) = (1-t)\sigma_0(s) + t\sigma_1(s) \in X$ (since X is convex). In particular, given a loop σ based at x_0 then $\sigma \sim \varepsilon_{x_0}$ and so $\pi_1(X,x_0) = \{[\varepsilon_{x_0}]\} \cong I$, the trivial group.

3. '$\Leftarrow$': Suppose that X is simply connected. Then, given paths σ_0 and σ_1 from x_0 to x_1, $\sigma_0 * \overline{\sigma}_1$ is a loop based at x_0. Hence $\sigma_0 * \overline{\sigma}_1 \sim \varepsilon_{x_0}$ or, equivalently, $[\sigma_0][\overline{\sigma}_1] = [\varepsilon_{x_0}]$. Hence, $[\sigma_0][\overline{\sigma}_1][\sigma_1] = [\varepsilon_{x_0}][\sigma_1]$. But $[\sigma_0][\overline{\sigma}_1][\sigma_1] = [\sigma_0][\varepsilon_{x_1}] = [\sigma_0]$ and $[\varepsilon_{x_0}][\sigma_1] = [\sigma_1]$ and so $[\sigma_0] = [\sigma_1]$, i.e. $\sigma_0 \sim \sigma_1$.

'$\Rightarrow$': Suppose that that all paths from x_0 to x_1 are equivalent. Let τ be a path from x_0 to x_1. Then given a loop σ at x_0, $\sigma * \tau$ is a path from x_0 to x_1 and so $\sigma * \tau \sim \tau$ or, equivalently, $[\sigma][\tau] = [\tau]$. Hence, $[\sigma][\tau][\overline{\tau}] = [\tau][\overline{\tau}]$. But $[\sigma][\tau][\overline{\tau}] = [\sigma][\varepsilon_{x_0}] = [\sigma]$ and $[\tau][\overline{\tau}] \sim [\varepsilon_{x_0}]$ so that $[\sigma] = [\varepsilon_0]$. Hence $\pi_1(X,x_0) = \{[\varepsilon_{x_0}]\}$, the trivial group, and so X is simply connected.

4. '$\Leftarrow$': Suppose that $\pi_1(X)$ is abelian. Suppose that ρ_0 and ρ_1 are paths from x_0 to x_1 and σ is a loop at x_0. Then $u_{\rho_0}([\sigma]) = u_{\rho_1}([\sigma]) \Leftrightarrow [\overline{\rho}_0][\sigma][\rho_0] = [\overline{\rho}_1][\sigma][\rho_1] \Leftrightarrow [\sigma][\rho_0][\overline{\rho}_1] = [\rho_0][\overline{\rho}_1][\sigma]$ (using $[\overline{\rho}_i] = [\rho_i]^{-1}$). But this final equality is true since $[\rho_0][\overline{\rho}_1]$ and $[\sigma] \in \pi_1(X,x_0)$ which is abelian. Hence $u_{\rho_0}([\sigma]) = u_{\rho_1}([\sigma])$ for all $[\sigma] \in \pi(X,x_0)$ and so $u_{\rho_0} = u_{\rho_1}$.

'$\Rightarrow$': Suppose that all paths from x_0 to x_1 induce the same isomorphism and that σ and τ are loops based at x_0. [*It may not be very clear how to* *proceed but to use the data we have to find two paths from x_0 to x_1. One exists since X is path-connected and so this suggests using one of the loops to obtain a second path and then applying the resulting isomorphisms to the other loop. This works.*] Let ρ be a path in X from x_0 to x_1. Then $\tau * \rho$ is a second path from x_0 to x_1 so that $u_{\tau*\rho}([\sigma]) = u_\rho([\sigma])$. This gives us $([\tau][\rho])^{-1}[\sigma]([\tau][\rho]) = [\rho]^{-1}[\sigma][\rho]$ which gives $[\rho]^{-1}[\tau]^{-1}[\sigma][\tau][\rho] = [\rho]^{-1}[\sigma][\rho]$ which gives $[\sigma][\tau] = [\tau][\sigma]$ proving that $\pi_1(X)$ is abelian.

5. (a) This is true. Given $[y] \in \pi_0(Y)$ then $y = f(x)$ for some $x \in X$ and $[y] = f_*([x])$.

(b) This is false. For example the inclusion map $i\colon \{0,1\} \to [0,1]$ is an injection but the first space has two path-components whereas the second has only one.

(c) This is false. For example the function $f\colon [0,1) \cup \{2\} \to [0,1]$ given by $f(x) = x$ for $x \in [0,1)$ and $f(2) = 1$ is a continuous bijection but the first space has two path components whereas the second has only one.