Spring Semester 2020–2021

MATH31052 *Topology*

Solutions 8

1. Let $\sigma\colon I \to S^1$ be the loop given by $\sigma(s) = \exp(2\pi is)$. Then this has the lift $\widetilde{\sigma}\colon I \to \mathbb{R}$ with $\widetilde{\sigma}(0)$ given by $\widetilde{\sigma}(s) = s$ and so $\phi([\sigma]) = \deg(\sigma) = \widetilde{\sigma}(1) = 1$. The loop $f\circ\sigma$ is given by $f\circ\sigma(s) = \exp(2\pi iks$ with lift $\widetilde{f\circ\sigma}(s) = ks$ so that $\phi\big(f_*([\sigma])\big) = \deg(f\circ\sigma) = k$. Thus the homomorphism $\mathbb{Z} \xrightarrow{\phi^{-1}} \pi(S^1, 1) \xrightarrow{f_*} \pi(S^1, 1) \xrightarrow{\phi} \mathbb{Z}$ maps $1 \mapsto k$ and so is given by $n \mapsto nk$.

2. Suppose that $\mathbf{x}$, $\mathbf{x}' \in \mathbb{R}^n$. Then, for $\mathbf{a} \in A$, $d(\mathbf{x}, A) \leqslant |\mathbf{x}-\mathbf{a}| \leqslant |\mathbf{x}-\mathbf{x}'| + |\mathbf{x}'-\mathbf{a}|$ (by the triangle inequality) and so $d(\mathbf{x}, A) - |\mathbf{x}-\mathbf{x}'| \leqslant |\mathbf{x}'-\mathbf{a}|$ for all $\mathbf{a} \in A$ so that $d(\mathbf{x}, A) - |\mathbf{x}-\mathbf{x}'| \leqslant d(\mathbf{x}', A)$ or, equivalently, $d(\mathbf{x}, A) - d(\mathbf{x}', A) \leqslant |\mathbf{x}-\mathbf{x}'|$. Similarly (interchanging $\mathbf{x}$ and $\mathbf{x}'$) $d(\mathbf{x}', A) - d(\mathbf{x}, A) \leqslant |\mathbf{x}-\mathbf{x}'|$ and so $|d(\mathbf{x}, A) - d(\mathbf{x}', A)| \leqslant |\mathbf{x}-\mathbf{x}'|$. Now, given $\varepsilon > 0$, if we put $\delta = \varepsilon$, $|\mathbf{x}-\mathbf{x}'| \leqslant \delta \Rightarrow |d(\mathbf{x}, A) - d(\mathbf{x}', A)| \leqslant \varepsilon$ and so $\mathbf{x} \mapsto d(\mathbf{x}, A)$ is continuous.

3. (a) This is false. For example $f = p\colon \mathbb{R} \to S^1$ given by $p(x) = \exp(2\pi ix)$ is a continuous surjection but $\pi_1(\mathbb{R}) \cong \{e\}$, the trivial group (by Problems 7, Question 2) whereas $\pi(S^1) \cong \mathbb{Z}$ and so the homomorphism f_* cannot be an epimorphism (surjection).

(b) This is false. For example the inclusion map $i\colon S^1 \to D^2$ is an injection but $\pi_1(S^1) \cong S^1$ and $\pi_1(D^2) \cong \{e\}$ and so the homomorphism f_* cannot be a monomorphism (injection).

(c) This is false. For example the restriction of the map p in (a) gives a continuous bijection $f\colon [0, 1) \to S^1$ providing a counterexample for the same reason as (a).

4. Suppose that $r\colon X \to A$ is a retraction. Then $r \circ i = \mathrm{id}_A\colon A \to A$ and so, by the functorial properties of the fundamental group, for $a_0 \in A$, $r_* \circ i_* = (r\circ i)_* = (\mathrm{id}_A)_* = \mathrm{id}_{\pi_1(A,a_0)}\colon \pi_1(A, a_0) \to \pi_1(X, a_0) \to \pi_1(A, a_0)$, i.e. $r_* \circ i_*(\alpha) = \alpha$ for all $\alpha \in \pi_1(A, a_0)$.

(a) To see that i_* is a monomorphism, suppose that $\alpha \in \pi_1(A, a_0)$ is such that $i_*(\alpha) = e \in \pi_1(X, a_0)$. Then $\alpha = r_* \circ i_*(\alpha) = r_*(e) = e \in \pi_1(A, a_0)$ and so $\ker(i_*) = \{e\}$, the trivial group which implies that the homomorphism i_* is a monomorphism.

(b) To see that r_* is an epimorphism, suppose that $\alpha \in \pi(A, a_0)$. Then $r_*\big(i_*(\alpha) = r_* \circ i_*\alpha) = \alpha$ and so r_* is a surjective homomorphism, i.e. an epimorphism.

5. The function is a homomorphism by Theorem 6.22. To see that it is an isomorphism we write down the inverse. Given a loop σ_1 in X based at x_0 and a loop σ_2 in Y based at y_0 then we may define a loop σ in $X \times Y$ based at (x_0, y_0) by $\sigma(s) = \big(\sigma_1(s), \sigma_2(s)\big)$. Then $([\sigma_1], [\sigma_2]) \mapsto [\sigma]$ is well-defined and provides the necessary inverse. Hence the given function is an isomorphism of groups.

6. From the result of Question 4, $\pi_1(S^1 \times S^1) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}$. Similarly, $\pi_1(S^1 \times I) \cong \pi_1(S^1) \times \pi_1(I) \cong \mathbb{Z} \times \{e\} \cong \mathbb{Z}$.

7. If $\mathbf{x} \in D^2 \setminus S^1$, there exists a retraction $r\colon D^2 \setminus \{\mathbf{x}\} \to S^1$ by projecting away from $\mathbf{x}$ (i.e. define $r(\mathbf{x}') = \mathbf{x} + t(\mathbf{x}' - \mathbf{x}) \in S^1$ for some unique $t \geqslant 0$: t is the non-negative root of a quadratic equation whose coefficients depend on the coordinates of $\mathbf{x}$ and $\mathbf{x}'$ and so is a continuous function of $\mathbf{x}'$ since the roots of a quadratic depend continuously on the coefficients). Then $r \circ i = \mathrm{id}_{S^1} : S^1 \to S^1$ and so, by the functorial properties of the fundamental group, $\pi_1(S^1, 1) \xrightarrow{i_*} \pi_1(D^2 \setminus \{\mathbf{x}\}, 1) \xrightarrow{r_*} \pi(S^1, 1)$ is the identity since $r_* \circ i_* = (r \circ i)_* = (\mathrm{id}_{S^1})_* = \mathrm{id}_{\pi_1(S^1,1)}$. But $\pi_1(S^1, 1) \cong \mathbb{Z}$ and so $\pi_1(D^2 \setminus \{\mathbf{x}\}, 1)$ is a non-trivial group and so $D^2 \setminus \{\mathbf{x}\}$ is not simply connected.
On the other hand, if $\mathbf{x} \in S^1$, then $D^2 \setminus \{\mathbf{x}\}$ is a convex subset of $\mathbb{R}^2$ and so is simply connected. Hence the set of points $\mathbf{x} \in D^2$ such that $D^2 \setminus \{\mathbf{x}\}$ is simply connected is S^1.
Suppose now that $f\colon D^2 \to D^2$ is a homeomorphism and $\mathbf{x} \in S^1$. Then the restriction $f\colon D^2 \setminus \{\mathbf{x}\} \to D^2 \setminus \{f(\mathbf{x})\}$ is a homeomorphism (since the restriction of f^{-1} gives the inverse) and so $D^2 \setminus \{\mathbf{x}\}$ simply connected implies that $D^2 \setminus \{f(\mathbf{x})\}$ is simply connected so that $f(\mathbf{x}) \in S^1$. Similarly $f^{-1}(S^1) \subset S^1$ and so $f(S^1) = S^1$. [Notice how this generalizes the cut-point arguments in the first section of the course.]