Three hours

THE UNIVERSITY OF MANCHESTER

ALGEBRAIC TOPOLOGY

28 May 2014

14:00 — 17:00

Answer **ALL** FOUR questions in Section A (40 marks in total). Answer **THREE** of the FOUR questions in Section B (45 marks in total). Answer **ALL** THREE questions in Section C (50 marks in total). If more than THREE questions from Section B are attempted then credit will be given for the best THREE answers.

Electronic calculators are permitted, provided they cannot store text.

SECTION A

Answer **ALL** FOUR questions.

A1. (a) Define what is meant by a *geometric simplicial surface.*

[The notions of *triangle* in $\mathbb{R}^n$, and its *vertices* and *edges* may be used without definition.]

(b) Define what is meant by the statement that a simplicial surface is *orientable.*

(c) Define what is meant by the statement that the orientability of a simplicial surface is a *topological invariant.*

[10 marks]

A2. (a) Define what it meant by a *geometric simplicial complex* K and its underlying space $|K|$.

[The notions of *geometric simplex* and *face* of a simplex may be used without definition.]

(b) An abstract simplicial complex has vertices $\{v_1, v_2, v_3, v_4\}$ and simplices $\{v_1, v_2\}$, $\{v_1, v_3\}$, $\{v_1, v_4\}$ and $\{v_2, v_3, v_4\}$ and their faces. Draw a realization K of this simplicial complex as a geometric simplicial complex in $\mathbb{R}^2$.

(c) Define the *Euler characteristic* of a simplicial complex and calculate the Euler characteristic of the simplicial complex in part (b).

(d) Draw the *first barycentric subdivision* K' of the geometric simplicial complex K in part (b).

(e) Find the Euler characteristic of K'.

[10 marks]

A3. (a) Define what is meant by the *r-chain group* $C_r(K)$, the *r-cycle group* $Z_r(K)$, and the *r-boundary group* $B_r(K)$ of a simplicial complex K.

(b) Write down without proof generators for the groups $Z_1(K)$ and $B_1(K)$ of the simplicial complex K in Question A2(b). Hence find the first homology group $H_1(K)$.

[10 marks]

A4. (a) Let K be the simplicial complex $\bar{\Delta}^7$ consisting of all of the faces of the standard 7-simplex in $\mathbb{R}^8$ (including the 7-simplex itself). Write down the simplicial homology groups of K, indicating a justification for your statement.

(b) Let L be the 3-skeleton of K. Calculate the Euler characteristic of L and hence, or otherwise, find its simplicial homology groups.

[10 marks]

SECTION B

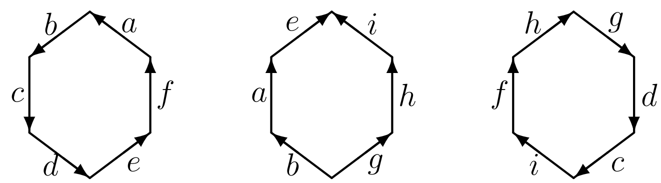
Answer **THREE** of the FOUR questions.

If more than THREE questions are attempted then credit will be given for the best THREE answers.

B5. (a) Explain how a *symbol* may be used to represent a closed surface arising from the identification in pairs of the edges of a polygon.

(b) State the classification theorem for closed surfaces in terms of symbols.

(c) The boundaries of three discs are identified as shown below.



Find a symbol for the resulting closed surface. By reducing the symbol to canonical form, or otherwise, identify the surface up to homeomorphism.

[15 marks]

B6. Let v_i be the ith standard basis vector in $\mathbb{R}^9$, $1 \leqslant i \leqslant 9$. Consider the set K of sixteen triangles with vertices v_i, v_j and v_k where ijk is one of

123, 124, 135, 145, 236, 247, 267, 347, 349, 357, 368, 389, 459, 579, 678, 789.

Verify that K is a simplicial surface, calculate its Euler characteristic and determine whether it is orientable. Identify the underlying space of K up to homeomorphism.

[15 marks]

B7. Describe a geometrical simplicial complex K whose underlying space is homeomorphic to the *Klein bottle* and indicate why it has this property.

Calculate the simplicial homology groups of K.

[15 marks]

B8. (a) Let K and L be simplicial complexes. Define what is meant by a *simplicial map* $|K| \to |L|$. Define what is meant by a *simplicial approximation* to a continuous map $f\colon |K| \to |L|$. Prove that a simplicial approximation to f is homotopic to f.

(b) Let K be the geometric simplicial complex in $\mathbb{R}$ with vertices 0, 1/2 and 1 and edges $\langle 0, 1/2\rangle$ and $\langle 1/2, 1\rangle$, so that the underlying space $|K|$ is the closed unit interval $[0, 1]$. Let $f\colon [0, 1] \to [0, 1]$ be the map $f(x) = x^2$. Prove that $f\colon |K| \to |K|$ does not have a simplicial approximation but that $f\colon |K'| \to |K|$ does have a simplicial approximation, where K' is the first barycentric subdivision of K.

[15 marks]

SECTION C

Answer **ALL** THREE questions.

C9. (a) Let p be an odd prime. What is a *p-symmetry* of a topological surface? What is meant by the statement that such a symmetry is *free*? Describe a free p-symmetry of the Klein bottle.

(b) Outline a proof that if there is a free p-symmetry of a surface S then there is a free p-symmetry of $S \# P_p$, the connected sum of S with a non-orientable surface of genus p.

[You may assume that, given a p-symmetry f of a surface S, there exists a non-empty open set U in S such that the sets $f^i(U)$, $0 \leqslant i \leqslant p-1$, are mutually disjoint.]

(c) Deduce that, if p divides $g-2$, then there is a free p-symmetry of P_g.

(d) Outline briefly a proof that there is no free p-symmetry of P_g if p does not divide $g-2$.

[17 marks]

C10. (a) Define what is meant by an *embedding* of a graph G in a closed topological surface S. When is such an embedding known as a 2-*cell embedding*? Outline a proof that, if a graph G with v vertices and e edges has a 2-cell embedding in a closed surface S with r regions, then

$$v - e + r = \chi(S), \text{ the Euler characteristic of } S.$$

(b) Assuming that $v - e + r \geqslant \chi(S)$ for a general embedding of a graph G with v vertices and e edges in a closed surface S with r regions (not necessarily 2-cell), prove that the average degree of a vertex satisfies

$$2e/v \leqslant 6(1 - \chi(S)/v).$$

(c) Define the *chromatic number* $k(G)$ of a graph G. Prove that if G may be embedded in S, a closed surfaced of Euler characteristic $\chi(S) \leqslant 0$, then

$$k(G) \leqslant N = [x], \text{ the integer part of the real number } x = \big(7 + \sqrt{49 - 24\chi(S)}\big)/2.$$

[Hint. Use the result of part (b) to prove that G must have a vertex of degree $\leqslant N-1$. You may find it helpful to observe that $x^2 - 7x + 6\chi(S) = 0$.]

[18 marks]

C11. (a) Define what is meant by saying that (X, A) is a *triangulable pair* of spaces.

(b) State the axioms for the reduced homology groups of triangulable spaces.

(c) Prove, from the axioms, that a homotopy equivalence of triangulable spaces $f\colon X \to Y$ induces isomorphisms $f_*\colon \tilde{H}_n(X) \to \tilde{H}_n(Y)$ of their reduced homology groups.

(d) Hence, show that the axioms determine the reduced homology groups of the n-sphere S^n for all $n \geqslant 0$.

[You may assume that the reduced homology groups of a point are all trivial.]

[15 marks]

END OF EXAMINATION PAPER