Autumn Semester 2017–2018

MATH41071/MATH61071 *Algebraic topology*

Abelian Groups

G.1 Definition. An *abelian group* is a non-empty set A together with a binary operation

$$+\colon A\times A\to A \quad (a_1,a_2)\mapsto a_1+a_2$$

such that

(i) $+$ is associative and commutative,

(ii) there is a (necessarily unique) element $0\in A$ such that $a+0=a$ for all $a\in A$,

(iii) given $a\in A$ there is a (necessarily unique) element $-a\in A$ such that $a+(-a)=0$.

G.2 Examples (a) $\mathbb{Z}$, the integers with the usual addition.

(b) $\mathbb{Z}_n=\{\,0,1,2,\ldots,n-1\,\}$, with addition modulo n.

(c) $0=\{\,0\,\}$, the trivial group.

(d) If A_1 and A_2 are abelian groups then so is $A_1\times A_2$ by

$$(a_1,a_2)+(a_1',a_2')=(a_1+a_1',a_2+a_2').$$

This is called the direct sum of the groups (and is sometimes denoted $A_1\oplus A_2$).

G.3 Definition. If A and B are abelian groups, a function $f\colon A\to B$ is a *homomorphism* if

$$f(a_1+a_2)=f(a_1)+f(a_2) \quad \text{for all } a_1,a_2\in A.$$

It follows that $f(0)=0$ and $f(-a)=-f(a)$.

The *kernel* of f is given by $\mathrm{Ker}(f)=\{\,a\in A\mid f(a)=0\,\}$.

The *image* of f is given by $\mathrm{Im}(f)=f(A)=\{\,f(a)\mid a\in A\,\}$.

These are examples of *subgroups*, i.e. subsets of a group which are themselves groups under the same binary operation.

If $\mathrm{Ker}(f)=0$, then f is a *monomorphism* (and this holds if and only if a homomrphism f is an injection).

If $\mathrm{Im}(f)=B$, then f is an *epimorphism.*

If f is both a monomorphism and an epimorphism then it is an *isomorphism* (and this occurs if and only if a homomorphism f is a bijection). In this case the inverse map $f^{-1}\colon B \to A$ is also an isomorphism. If such an f exists then we say that A and B are *isomorphic* and write $A \cong B$.

G.4 Definition. If $f\colon A \to B$ and $g\colon B \to C$ are homomorphisms then we say that the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is *exact* (at B) when $\mathrm{Im}(f) = \mathrm{Ker}(g)$. This means that $g \circ f = 0$ (equivalent to $\mathrm{Im}(f) \subset \mathrm{Ker}(g)$ and if $g(b) = 0$ then $b = f(a)$ for some $a \in A$ (equivalent to $\mathrm{Im}(f) \supset \mathrm{Ker}(g)$).

G.5 Examples. (a) $0 \to A \xrightarrow{f} B$ is exact if and only if f is a monomorphism.

$A \xrightarrow{f} B \to 0$ is exact if and only if f is an epimorphism.

$0 \to A \xrightarrow{f} B \to 0$ is exact if and only if f is an isomorphism. Notice that exactness of this sequence means that it is exact at both A and B.

(b) $0 \to A_1 \xrightarrow{f} A_1 \times A_2 \xrightarrow{g} A_2 \to 0$ is exact where $f(a_1) = (a_1, 0)$, $g(a_1, a_2) = a_2$.

(c) $0 \to \mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \mathbb{Z}_2 \to 0$ is exact where $f(n) = 2n$ and $g(n) = n$ (reduction modulo 2).

(d) A *short exact sequence* is an exact sequence of the form

$$0 \to A \to B \to C \to 0$$

which means that the seqence is exact at A, B and C. Examples (b) and (c) are short exact sequences.

(e) Given any homomorphism $f\colon A \to B$, the sequence

$$0 \to \mathrm{Ker}(f) \xrightarrow{i} A \xrightarrow{f} \mathrm{Im}(f) \to 0$$

is short exact where i is the inclusion map.

G.6 Definition. Given a subgroup B of a group A we may define an equivalence relation on A by $a_1 \sim a_2 \Leftrightarrow a_1 - a_2 \in B$. The equivalence classes of this relation are called the *cosets* of B in A.

Notice that the coset of an an element $a_0 \in A$ is given by $[a] = \{\, a \in A \mid a - a_0 \in B \,\} = \{\, b + a_0 \mid b \in B \,\} = B + a_0$.

The set of cosets of B in A is denoted A/B and may be made into an abelian group by the operation

$$(B + a_1) + (B + a_2) = B + (a_1 + a_2).$$

With this structure B/A is called the *quotient group*.

G.7 Example. If $A = \mathbb{Z}$ and $B = 2Z = \{ 2n \mid n \in Z \}$ then $A/B = \mathbb{Z}/2\mathbb{Z}$ has two elements: the set of even integers $2\mathbb{Z}$ and the set of odd integers $2\mathbb{Z} + 1$. Clearly $\mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}_2$ with the isomorphism $2\mathbb{Z} \mapsto 0$, $2\mathbb{Z} + 1 \mapsto 1$.

G.8 Proposition. If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is short exact, then g induces an isomorphism

$$\overline{g}\colon B/f(A) \to C$$

by $\overline{g}(f(A) + b) = g(b)$.

Proof. Exercise. □

Applying this result to the short exact sequence of Example G.5(e) gives an important result.

G.9 Corollary [The first isomorphism theorem]. Given any homomorphism $f\colon A \to B$, f induces an isomorphism

$$\overline{f}\colon A/\mathrm{Ker} f \to \mathrm{Im} f$$

by $\overline{f}(\mathrm{Ker} f + a) = f(a)$.

Another result concerning isomorphic quotient groups is useful in the calculation of simplicial homology groups.

G.10 Proposition [The second isomorphism theorem]. If B and C are subgroups of an abelian group A, then the inclusion map $B \to B + C$ induces an isomorphism

$$B/(B \cap C) \cong (B + C)/C.$$

(Here $B + C = \{ b + c \mid b \in B, c \in C \}$.

Proof. Exercise. □

G.11 Definition. An abelian group A is *finitely generated* if there is a finite set of elements $a_1, a_2, \ldots, a_r \in A$ such that every element $a \in A$ can be expressed in the form $a = \sum n_i a_i$ for $n_i \in \mathbb{Z}$.

If $r = 1$ and there is a single generator then A is *cyclic* and either $A \cong \mathbb{Z}$ or $A \cong \mathbb{Z}_n$ for some n (the least positive integer such that $na = 0 \in A$, the *order* of the element — see Definition G.13 below).

If $\sum n_i a_i = 0$ if and only if $n_i = 0$ for all i then A is *freely generated* by $a_1, a_2 \ldots a_r$ and is *free abelian.* In this case it can be shown that the number r is well-defined (cf. the dimension of a vector space); this number is called the *rank* of A.

Given a free group A of rank r freely generated by $a_1, a_2, \ldots, a_r$, then a homomorphism $f\colon A \to B$ may be determined by assigning arbirary values to $f(a_1), f(a_2), \ldots f(a_r)$, for $f(\sum n_i a_i) = \sum n_i f(a_i)$.

G.12 Proposition. A free group A freely generated by $a_1, a_2, \ldots, a_r$ is isomorphic to $\mathbb{Z}^r$; an isomorphism $f\colon A \to \mathbb{Z}^r$ is given by $f(a_i) = e_i$.

Proof. Exercise. □

G.13 Definition. Given an abelian group A, an element $a \in A$ is called a *torsion element* if $na = 0 \in A$ for some positive integer n. For $a \neq 0$, the least such n is called the *order* of the element.

G.14 Proposition. Given an abelian group A, the subset of torsion element forms a subgroup $T(A)$ called the *torsion subgroup.* The quotient group $A/T(A)$ is a free group. If A is finitely generated, then so is $A/F(A)$. The *rank of* A is defined to be the rank of $A/F(A)$.

Proof. Exercise. □

G.15 Proposition. $\mathbb{Z}_m \times Z_n \cong \mathbb{Z}_{mn}$ if and only if m and n are coprime. If m and n are coprime then there is an isomorphism $\mathbb{Z}_m \times Z_n \cong \mathbb{Z}_{mn}$ given by $(i, j) \mapsto ni + mj$.

Proof. Exercise. □

G.16 Theorem [Classification theorem for finitely generated abelian groups]. Every finitely generated abelian group is isomophic to a unique group of the form

$$\mathbb{Z}^r \times \mathbb{Z}_{\lambda_1} \times \mathbb{Z}_{\lambda_2} \times \cdots \times \mathbb{Z}_{\lambda_s}$$

where $r \geqslant 0$, $s \geqslant 0$ and λ_i divides λ_{i+1} for each i.

Proof. Omitted. See for example B. Hartley and T.O. Hawkes, *Rings, modules and linear algebra,* Chapman and Hall (1970), chapters 7 and 10. $\square$

G.17 Remark. An isomorphism $A \cong \mathbb{Z}^r \times \mathbb{Z}_{\lambda_1} \times \mathbb{Z}_{\lambda_2} \times \cdots \times \mathbb{Z}_{\lambda_s}$ restricts to an isomorphism of the torsion subgroups $T(A) \cong \{0\} \times \mathbb{Z}_{\lambda_1} \times \mathbb{Z}_{\lambda_2} \times \cdots \times \mathbb{Z}_{\lambda_s} \cong \mathbb{Z}_{\lambda_1} \times \mathbb{Z}_{\lambda_2} \times \cdots \times \mathbb{Z}_{\lambda_s}$ and induces an isomorphism $A/T(A) \cong \mathbb{Z}^r$. So r is the rank of A. The numbers λ_1, λ_2, ..., λ_s are called the *torsion coefficients* of A.

G.18 Theorem. If $0 \to A \to B \to C \to 0$ is a short exact sequence of finitely generated abelian groups then

$$\mathrm{rank}(B) = \mathrm{rank}(A) + \mathrm{rank}(C).$$

Proof. Omitted. See for example P.J. Giblin, *Graphs, surfaces and homology,* Chapman and Hall (1977) appendix on abelian groups. $\square$