Problems 4: Simplicial complexes

1. Prove that if \( x \in \langle v_0, v_1, \ldots, v_r \rangle \) is a point in an \( r \)-simplex then \( x \) can be written uniquely in the form \( x = \sum_{i=0}^{r} t_i v_i \) where \( \sum_{i=0}^{r} t_i = 1 \). [Proposition 4.9]

2. Prove that an isomorphism \( f : K_1 \to K_2 \) of geometric simplicial complexes induces a homeomorphism \( |f| : |K_1| \to |K_2| \) of their underlying spaces. [Corollary 4.11]

3. Describe triangulations of the closed cylinder \( I^2/(s,0) \sim (s,1) \) and of the Möbius band \( I^2/(s,0) \sim (1-s,1) \).

4. The dunce hat is obtained by identifying all three sides of a triangle as shown. Construct a simplicial complex which triangulates this space.

5. Given a simplicial complex \( K \) with \( n_r \) \( r \)-simplices for \( 0 \leq r \leq \dim K \), the Euler characteristic \( \chi(K) \) of \( K \) is defined by

\[
\chi(K) = \sum_{r=0}^{\dim K} (-1)^r n_r.
\]

Let \( s_n = \langle v_0, v_1, \ldots, v_n \rangle \), an \( n \)-simplex. Find the Euler characteristic of the following simplicial complexes:

(i) \( \Delta^n \) (for \( n \geq 0 \));

(ii) \( \Delta^{[n-1]} \) (the \( (n-1) \)-skeleton of \( \Delta^n \), for \( n \geq 1 \));

(iii) \( \Delta^{[n-2]} \) (the \( (n-2) \)-skeleton of \( \Delta^n \), for \( n \geq 2 \));

(iv) the simplicial complex you constructed in Question 4.
6. Prove that the standard $n$-simplex $\Delta^n$ is homeomorphic to the $n$-ball $D^n$.

[Hint: Produce a sequence of homeomorphisms

$$\Delta^n \cong \{(t_1, \ldots, t_n) \in \mathbb{R}^n \mid t_i \geq 0, \sum_{i=1}^{n} t_i \leq 1\} \cong I^n = \{(t_1, \ldots, t_n) \mid t_i \geq 0, \max(t_i) \leq 1\}$$

$$\cong [-1, 1]^n = \{(t_1, \ldots, t_n) \mid t_i \geq 0, \max(|t_i|) \leq 1\} \cong D^n = \{(t_1, \ldots, t_n) \mid \sum_{i=1}^{n} t_i^2 \leq 1\}$$

or you may find something simpler.]