Coursework

This coursework will count 10% of the marks. There are 30 marks for the coursework from the two questions overleaf. All questions should be attempted. Marks will be given for the quality of the mathematical writing. The work should be handed in until 15:00 on Friday 17 November to the Teaching and Learning Office Reception or to my office 1.106a both in the Alan Turing Building or, alternatively, via email until 23:59 on the same day.

Family name: ....

other names: ....

Registration Number: ....

Degree programme: ....

Year of programme: ....

Course code: MATH41071 MATH61071
(encircle)

I confirm that the work submitted is my own work.

Signature: ....

Date: ....

1. (a) Let v_i be the ith standard basis vector in $\mathbb{R}^{10}$, $1 \leqslant i \leqslant 10$. Consider the set K of twenty triangles with vertices v_i , v_j and v_k where (i, j, k) is one of

$(1, 2, 7), (1, 2, 10), (1, 4, 7), (1, 4, 10), (2, 3, 5), (2, 3, 9), (2, 5, 10), (2, 7, 9), (3, 4, 5), (3, 4, 8),$

$(3, 6, 8), (3, 6, 9), (4, 5, 7), (4, 8, 10), (5, 6, 7), (5, 6, 10), (6, 7, 8), (6, 9, 10), (7, 8, 9), (8, 9, 10).$

Verify that K is a simplicial surface.

(b) Find a symbol which represents the simplicial surface K and reduce this to canonical form. Hence, identify the underlying space of K up to homeomorphism as a space in the classification theorem for closed surfaces.

(c) Confirm your answer to (b) by calculating the Euler characteristic of K and determining whether it is orientable.

[6+6+3=15 marks]

Solution

(a) (i) The intersection condition is fulfilled since the vertices are chosen to be linearly independent vectors in $\mathbb{R}^{10}$. Hence, for two triangles $\sigma_1 = \langle v_i, v_j, v_k\rangle$ and $\sigma_2 = \langle v_\ell, v_m, v_n\rangle$ the intersection is given by

$$\sigma_1 \cap \sigma_2 = \sigma_1 \cap V = \sigma_2 \cap V$$

with

$$V = \operatorname{span}(v_i, v_j, v_k) \cap \operatorname{span}(v_\ell, v_m, v_n) = \operatorname{span}(\{v_i, v_j, v_k\} \cap \{v_\ell, v_m, v_n\}).$$

Here, that latter equality holds because of the linear indpendence of the set $\{v_i, v_j, v_k, v_\ell, v_m, v_n\}$.

(ii) For the connectedness note, that there is a path

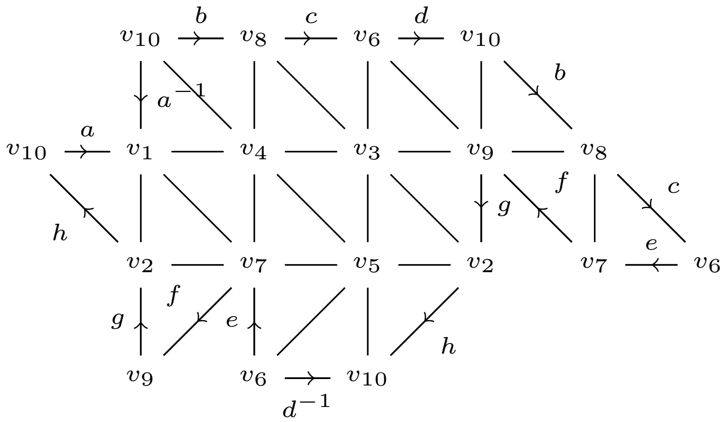
$$v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_7 - v_8 - v_9 - v_{10}$$

containing all vertices.

(iii) For the link condition we have to check all vertices

link(v_1): $v_2 - v_7 - v_4 - v_{10} - v_2$
link(v_2): $v_1 - v_7 - v_9 - v_3 - v_5 - v_{10} - v_1$
link(v_3): $v_2 - v_5 - v_4 - v_8 - v_6 - v_9 - v_2$
link(v_4): $v_1 - v_7 - v_5 - v_3 - v_8 - v_{10} - v_1$
link(v_5): $v_2 - v_3 - v_4 - v_7 - v_6 - v_{10} - v_2$
link(v_6): $v_3 - v_8 - v_7 - v_5 - v_{10} - v_9 - v_3$
link(v_7): $v_1 - v_2 - v_9 - v_8 - v_6 - v_5 - v_4 - v_1$
link(v_8): $v_3 - v_4 - v_{10} - v_9 - v_7 - v_6 - v_3$
link(v_9): $v_2 - v_3 - v_6 - v_{10} - v_8 - v_7 - v_2$
link(v_{10}): $v_1 - v_2 - v_5 - v_6 - v_9 - v_8 - v_4 - v_1$

[6 marks]



(b)

The corresponding symbol can be read off, e.g. by starting at the leftmost vertex with label v_{10} going around the polygon clockwise: $aa^{-1}bcdbcefghd^{-1}efgh$.

$$
\begin{aligned}
aa^{-1}\dot{b}cd\dot{b}cefghd^{-1}efgh &\sim aa^{-1}(bb)d^{-1}c^{-1}cefghd^{-1}efgh && 2.21(vi)\\
&\sim bbaa^{-1}\dot{d}^{-1}c^{-1}cefgh\dot{d}^{-1}efgh && 2.22(i)\\
&\sim bbaa^{-1}(d^{-1}d^{-1})h^{-1}g^{-1}f^{-1}e^{-1}c^{-1}cefgh && 2.21(vi)\\
&\sim bbd^{-1}d^{-1}(aa^{-1})h^{-1}g^{-1}f^{-1}e^{-1}c^{-1}cefgh && 2.22(i)\\
&\sim bbd^{-1}d^{-1}h^{-1}(g^{-1}f^{-1}e^{-1}c^{-1}cefg)h && 2.21(v)\\
&\sim bbd^{-1}d^{-1}(h^{-1}h)g^{-1}f^{-1}e^{-1}c^{-1}cefg && 2.22(ii)\\
&\sim bbd^{-1}d^{-1}g^{-1}(f^{-1}e^{-1}c^{-1}cef)g && 2.21(v)\\
&\sim bbd^{-1}d^{-1}g^{-1}gf^{-1}e^{-1}c^{-1}cef && 2.22(ii)\\
&\sim bbd^{-1}d^{-1}f^{-1}(e^{-1}c^{-1}ce)f && 2.21(v)\\
&\sim bbd^{-1}d^{-1}(f^{-1}f)e^{-1}c^{-1}ce && 2.22(ii)\\
&\sim bbd^{-1}d^{-1}e^{-1}(c^{-1}c)e && 2.21(v)\\
&\sim bbd^{-1}d^{-1}(e^{-1}e)(c^{-1}c) && 2.22(ii)\\
&\sim bbd^{-1}d^{-1} && 2.21(v)+2.21(v)\\
&\sim x_1x_1x_2x_2 && 2.21(i)+2.21(ii)
\end{aligned}
$$

Hence, the corresponding surface is P_2. [6 marks]

(c) Using the fact that for a simplicial surface we $2e = 3f$ one calculates $\chi(K) = v - e + f = 10 - 30 + 20 = 0$.

Moreover, assume there exists an orientation of all triangles, such that we have coherence along common edges. W.l.o.g we may assume that the traingle on the top left with vertices v_{10}, v_8 and v_4 is oriented as $\langle v_4, v_{10}, v_8\rangle$. This implies a clockwise orientation for all other triangles in the picture via coherence along common edges. Hence, the induced orientations for the edge between v_{10} and v_8 is $\langle v_{10}, v_8\rangle$ both for $\langle v_4, v_{10}, v_8\rangle$ and $\langle v_{10}, v_8, v_9\rangle$. A contradiction to the assumed coherence. Hence, K is not orientable.

This is in line with the finding of (b), since P_2 is not orientable and has Euler characteristic $2g - 2 = 0$. [3 marks]

Feedback: *In general the question was done well. Note that the solution for (b) given above follows strictly the algorithm from the proof of Theorem 2.20. There are shortcuts in this case and in general it is fine to use these shortcuts. But you should make sure that you really understood the algorithm and not just found a solution "by chance".*

One common mistake in (c) was to take the fact $|K| \cong P_2$ from (b) for granted and conclude that the surface is non-orientable. However, you were asked to confirm your finding of (b). Hence, you were not allowed to use this result, but had to determine the orientability type directly.

2. *A topological manifold with boundary* of dimension d is defined a Hausdorff, second countable topological space X, which is locally homeomorphic to $H^+ = \mathbb{R}^{d-1} \times [0, \infty)$, i.e. every $x \in X$ admits an open neighbourhood which is homeomorphic to an open subset of H^+. The *boundary* of X denoted as ∂X is the set of points, where X is not locally Euclidean (or equivalently the points in the image of the hyperplane $H = \mathbb{R}^{d-1} \times \{0\}$ under the local homeomorphisms).

Note, that with this definition a manifold X is also a manifold with boundary, where $\partial X = \emptyset$.

(a) Consider the simplicial complexes below and check whether their underlying space is a surface with boundary. In each case justify your answer.

(i) K_1 consisting of the triangles $\langle 0, e_1, e_2\rangle$ and $\langle 0, -e_1, e_2, \rangle$ and their faces, where $e_1,\ e_2 \in \mathbb{R}^2$ are the standard basis vectors;

(ii) K_2 consisting of the triangles $\langle 0, e_1, e_2\rangle$ and $\langle 0, -e_1, -e_2\rangle$ and their faces;

(iii) K' consisting of all but one of the triangles of a simplicial surfaces K and all faces of these triangles.

(b) Give a suitable definition of a *simplicial surface with boundary* by relaxing the link condition such that the corresponding underlying space is a path-connected compact surfaces with boundary. Show you definition works in this way by adapting the proof of Proposition 2.5.

[7+8=15 marks]

Solution

(a) (i) It is a surface with boundary. The underlying space coincides with het triangle $\langle -e_1, e_1, e_2\rangle$. First, note that

$$U_1 = \operatorname{int}\langle -e_1, e_1, e_2\rangle \cup \operatorname{int}\langle -e_1, e_1\rangle$$

is an open subset of H^+. Hence, it provides the required open neighbour for all its points. Similar

$$U_2 = \operatorname{int}\langle -e_1, e_1, e_2\rangle \cup \operatorname{int}\langle -e_1, e_2\rangle, \quad U_3 = \operatorname{int}\langle -e_1, e_1, e_2\rangle \cup \operatorname{int}\langle e_1, e_2\rangle.$$

give open subsets which are both homeomorphic to U_1 (which was open in H^+) via

$$e_2 \mapsto e_1; -e_1 \mapsto -e_1, e_1 \mapsto e_2; \qquad e_2 \mapsto -e_1; e_1 \mapsto e_1, -e_1 \mapsto e_2$$

and linear extension. Now, the claim follow, since $|K_1| = U_1 \cup U_2 \cup U_3$.

[2 marks]

(ii) This is not a surface with boundary. Indeed, for the point $(0,0)$ there does not exist an open neigbourhood, which is homeomorphic to an open subset of H^+. Assume to the contrary that

$$\phi\colon V \to U \subset H^+$$

is such a homeomorphism. Then after restriction of ϕ we may assume that $V = B_\epsilon(0) \cap |K_2|$. Then V and U are path-connected. Moreover, 0 is a cut-point of V. Hence, $y = \phi(0)$ has to be a cut-point of U. However, since U is open, for some $\epsilon > 0$ we have $B_\epsilon^{H^+}(y) \subset U$. Set $D = B_{\epsilon/2}^{H^+}(y)$. Now, since U is connected for every pair of point $y_1, y_2 \in U \setminus \{y\}$ there is a path $\sigma\colon I \to U$ connecting y_1 and y_2.

$$t_- = \min \sigma^{-1}(D); \qquad t_+ = \max \sigma^{-1}(D).$$

Assume $\sigma(t_\pm) = y + \frac{\epsilon}{2}(e^{i\alpha_\pm})$ (using the usual identification $\mathbb{R}^2 \cong \mathbb{C}$). Assume w.l.o.g $\frac{3}{2}\pi \geqslant \alpha_+ \geqslant \alpha_- \geqslant -\frac{1}{2}\pi$ (else we switch to the reverse path). Then the following gives a path in $U \setminus \{y\}$ between y_1 and y_2

$$\sigma'(t) = \begin{cases} \sigma(t) & t \notin (t_-, t_+) \\ y + \frac{\epsilon}{2} e^{\left(\frac{t-t_-}{t_+-t_-}\alpha_+ + (1-\frac{t-t_-}{t_+-t_-})\alpha_-\right)i} & t \in [t_-, t_+]. \end{cases}$$

It's easier to describe what this path does in words, than to write down the formula: it follows the original path σ until it enters the compact subset $D \subset B_\epsilon^{H^+}(y)$. Then it follows the "curved" boundary of D to the point where σ leaves D for the last time. Then it again follows σ. [2 marks]

(iii) The underlying space is again a surface with boundary. Assume the removed triangle is $\langle v_0, v_1, v_2\rangle$. At points outside the removed triangle $|K'|$ is still locally Euclidean. By translation we may assume that the corresponding open subset of $\mathbb{R}^2$ lies even in H^+.

It remains to check points on the edges and vertices of the removed triangle. Assume $x = v_0$ or x lies on the interior of $\langle v_0, v_1\rangle$. Then by the link condition for K there are pairwise distinct triangles $\langle v_0, v_1, v_2\rangle \ldots \langle v_0, v_{\ell-1}, v_\ell\rangle$ with $v_1 = v_\ell$. After renoving $\langle v_0, v_1, v_2\rangle$ we may map the union of the remaining triangles homeomorphic to the union of the trianges

$$\langle 0, e^0, e^{\frac{i\pi}{\ell-2}}\rangle, \ldots, \langle 0, e^{\frac{(\ell-2)i\pi}{\ell-2}}, e^{i\pi}\rangle.$$

via linear extension, where $v_0 \mapsto 0$ and $v_{2+k} \mapsto e^{\frac{ki\pi}{\ell-1}}$ for $0 \leqslant k \leqslant \ell-2$. By this the union of the triangles $\langle v_0, v_{k-1}, v_k\rangle$ minus the edges $\langle v_{k-1}, v_k\rangle$ is mapped to the union of triangles $\langle 0, e^{\frac{(k-1)i\pi}{\ell-2}}, e^{ki\pi}\rangle$ without the edges $\langle e^{\frac{(k-1)i\pi}{\ell-2}}, e^{ki\pi}\rangle$. The first union is an open subset of $|K'|$ which contains x and the latter one is an open subset in H^+. [3 marks]

(b)

Definition 1. A *simplicial surface with boundary* is a finite set K of triangles in some $\mathbb{R}^n$, such that the following conditions are fulfilled

intersection condition: every two triangles are disjoint or intersect in a common face;

connectedness: every pair of vertices is connected via a path of edges of triangles in K;

link condition: for every vertex v the link is simple path (possibly closed, but without self-crossings).

We claim that the underlying space is a compact, path-connected surface with boundary.

[3 marks]

Proof. The proof for second-countability, Hausdorff property, compactness and path-connectedness is literally the same as in Proposition 2.20. It remains to show that the underlying space is locally ismorphic to $H^+ = \mathbb{R} \times [0, \infty)$.

Let x be in the interior of a triangle σ then this triangle is homeomorphic to $\langle 0, e_1, e_2 \rangle$, where interiors are mapped to interiors. Hence, the interior of σ provides an open naighbourhood of x which is homeomorphic to an open subset of H^+.

Note, that it follows from the new link condition that every edge is contained in at most two triangles. Moreover, it is contained in only one triangle iff the link of its vertices is not closed. Now, let x be an interior point of an edge. If this edge is contained in two triangles, then we can map the union of both triangles via linear extension to $\langle 0, e_1, e_2 \rangle \cup \langle 0, -e_1, e_2 \rangle$, such that the image of x lies in the interior of the edge $\langle 0, e_2 \rangle$. The union of the interiors of both triangles and the interior of the common edge will be mapped to $\operatorname{int}\langle -e_1, e_1, e_2 \rangle$ which is an open subset of H^+.

Consider a vertex v which has a non-closed simple path as its link. Assume that $x = v$ or x lies in an interior of an edges containing v. The we may use the same homeomorphism as in 2(a)(iii) to map the open subset $V \subset |K|$ to an open subset of H^+, where

$$V = \left(\bigcup_{v \in \sigma \in K} \sigma \right) \setminus \left(\bigcup_{\tau \in \operatorname{link}(v)} \tau \right).$$

□

[5 marks]

Feedback: *The question was challenging. I guess you have been rarely asked to come up with a definition yourself. However, this is actually what mathematicians in their research and the tricky point is to come up with a* good *definition which matches the thing/property..., which you want to describe.*

Most people attempted the question and where able to secure at least partial mark. Some of the solutions were actually very good. It helped to draw picture of the situation. Pictures can also help to explain an argument, but they cannot replace the argument in a (written) proof.

Most people hat the correct intuition which of the examples are locally homeomorphic to H^+. However, the solutions were often lacking a precise argument. Indeed, it was important to refer to the actual definition of being locally homeomorphic to H^+: For every choice of an $x \in |K|$ you needed to state an open neighbourhood $x \in V \subset |K|$, an open subset $U \subset H^+$ and a homeomorphism $\phi\colon V \to U$. I didn't insist of seeing an exact description of the homeomorphism in coordinates, but I wanted to see a description of a construction which made me believe that you could write down everything in coordinates if you are forced to.

Many tried to use open subsets of the form $B_\epsilon(x) \cap |K|$. This does work for example (i) and (2), but for $(a)(iii)$ and (b) this doesn't work anymore. Here, it's better to use the interiors of triangles and edges to find open subset. The big advantage is that we can use our linear extension construction (which we now call simplicial maps) to get homeomorphisms to subsets of $\mathbb{R}^2$ (or H^+) in an easy way (this is the power of linear algebra!).

A nice observation by one of you was that a triangle minus a vertex is itself homeomorphic to H^+. However, it's not so easy to come up with an construction of a homeomorphism (radial projection + a kind of stereographic projection would for example work).

One common mistake was to forget that we are only looking for a local *homeomorphism. Hence, a global homeomorphism to H^+ usually does not exists. Indeed, for example in 2(a)(ii) it was not sufficient to say that H^+ does not have a cut point, but $|K_2|$ has 0 as a cut-point, since we never proved that a space with a cut point cannot even locally be homoemorphic to one without. What you had to show (or at least mention) first, was that every connected open subset of H^+ does not have a cut point.*

Another misconception was to speak about points on the boundary of $|K|$ while proving that it is a surface with boundary. Note, that boundary *becomes a meaning once you proved that you have a manifold with boundary. There is also a* different *concept of boundary for subsets $Y \subset X$ of a topological space X: all points such that every open neighbourhood contains points of Y and $X \setminus Y$. However, this is not invariant under homeomorphisms, but depend on the embedding. For example, the boundary in this sense of a triangle in $\mathbb{R}^3$ is the whole triangle (this was not our definition in the lecture/notes).*

In order to come up with the correct definition it was a good idea to look at the example in 2(a) and check what kind of links occur there and in which sense the link of 0 in the counterexample (a)(iii) differs from the links observed in (a)(i) and (a)(ii). This should give you at least a guess for the correct condition. I have seen some definitions from you which were different but equivalent to the one given above. This is absolutely fine. The harder part was to prove that the guess was correct. A good idea was to follow closely the proof of Theorem 2.5, which already gives the required open neighbourhoods and homeomorphisms for some cases.