SECTION A

Answer **ALL** FOUR questions.

A1.

(a) Define what is meant by a *topological manifold*.

(b) Give an example of a topological space which is not a manifold and state which of the defining properties is not fulfilled.

(c) State the classification theorem for connected compact topological surfaces.

[10 marks]

Solution

(a) Let n be a non-negative integer. An *n-dimensional (topological) manifold* is a topological space X which

(i) is Hausdorff,

(ii) is second countable (i.e. has a countable basis), and

(iii) is *locally Euclidean*, i.e. each point $x \in X$ lies in an open subset V in X which is homeomorphic to an open subset $U \subset \mathbb{R}^n$ (with the usual topology).

[5 marks, bookwork]

(b) There are plenty of examples. Just to name two of them

(i) The union of two lines in $\mathbb{R}^2$ intersecting in a point (e.g. $\{(x, y) \in \mathbb{R}^2 \mid x = 0 \text{ or } y = 0\}$), which is not locally Euclidean at the intersection point.

(ii) $\mathbb{R} \times \{0, 1\}/\sim$, with $(x, 0) \sim (x, 1)$ for $x \neq 0$. This space is not Hausdorff.

[2 marks, bookwork]

(c) Every connected compact topological surface (or *closed surface*) is homeomorphic to one and only one of:

(i) S^2,

(ii) T_g for some $g \geqslant 1$ (where $T_1 = S^1 \times S^1$ and $T_{g+1} = T_g \# T_1$ for $g \geqslant 1$),

(iii) P_g for some $g \geqslant 1$ (where $P_1 = P^2$ and $P_{g+1} = P_g \# P_1$ for $g \geqslant 1$). [3 marks, bookwork]

[**Total: 10 marks**]

Feedback: The question was meant to be easy. The definition of a manifold was fundamental for the first part of the course (on surfaces). Some people forgot to explain what locally Euclidean means in part (b). People couldn't come up with a counterexample in part (b). Part (c) was generally done well.

A2.

(a) Define what is meant by a *geometric simplicial complex* K.
[The notions of geometric simplex and face of a simplex may be used without definition.]

(b) What is the *underlying space* $|K|$ of such a simplicial complex K?

(c) An abstract simplicial complex has vertices v_1, v_2, v_3, v_4, v_5 and simplices $\{v_1, v_2, v_3\}$, $\{v_2, v_4\}$, $\{v_4, v_5\}$, $\{v_3, v_5\}$, $\{v_2, v_5\}$ and their faces. Draw a realisation K of this simplicial complex as a geometric simplicial complex in $\mathbb{R}^2$.

(d) Define the *Euler characteristic* of a simplicial complex and calculate the Euler characteristic of the simplicial complex in part (c).

(e) Draw the first barycentric subdivision K' of the geometric simplicial complex K in part (c).

(f) Find the Euler characteristic of K'.

[10 marks]

Solution

(a) A *(geometric) simplicial complex* is a non-empty finite set K of simplices in some Euclidean space $\mathbb{R}^n$ such that

  (a) **the face condition:** if $\sigma \in K$ and $\tau \prec \sigma$ then $\tau \in K$,

  (b) **the intersection condition:** if σ_1 and $\sigma_2 \in K$ then $\sigma_1 \cap \sigma_2 = \emptyset$ or $\sigma_1 \cap \sigma_2 \prec \sigma_1$, $\sigma_1 \cap \sigma_2 \prec \sigma_2$.

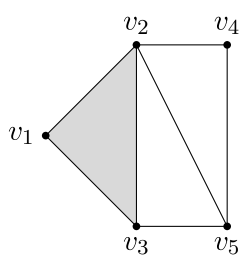
[2 marks, bookwork]

(b) The *underlying space* $|K|$ of a simplicial complex K is given by

$$|K| = \bigcup_{\sigma \in K} \sigma \subset \mathbb{R}^n$$

with the subspace topology. [1 mark, bookwork]

(c) A realisation is given by the following picture



[2 marks, similar to question set]

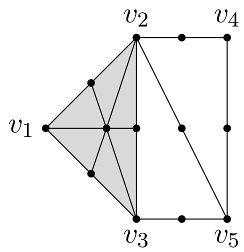
(d) The *Euler characteristic* of a simplicial complex K is given by the alternating sum

$$\chi(K) = \sum_{r=0}^{\infty} (-1)^r n_r$$

where n_r is the number of simplices of dimension r. In this case $\chi(K) = 5 - 7 + 1 = -1$.

[2 marks, bookwork]

(e) The barycentric subdivision is given by the following picture



[2 marks, similar to question set]

(f) The Euler characteristic is again -1, since barycentric subdivisions does not change the Euler characteristic [It can also be found by counting simplices.] [1 mark, simple application]

[Total: 10 marks]

Feedback: The question was generally done well. Occasionally people mixed up the definitions of simplicial complex with that of a simplicial surface (no link condition, no connectedness was needed here)

A3.

(a) Define what is meant by the r-*chain group* $C_r(K)$, the r-*cycle group* $Z_r(K)$, and the r-*boundary group* $B_r(K)$ of a simplicial complex K.

(b) Write down, without proof, generators for the groups $Z_1(K)$ and $B_1(K)$ of the simplicial complex K in Question A2(c). Hence, find the first homology group $H_1(K)$.

[10 marks]

Solution

(a) For $r \in \mathbb{Z}$. the r-chain group of K, denoted $C_r(K)$, is the free abelian group generated by K_r, the set of oriented r-simplices of K subject to the relation $\sigma + \tau = 0$ whenever σ and τ are the same simplex with the opposite orientations. [2 marks, bookwork]

For each $r \in \mathbb{Z}$ we define the boundary homomorphism $d_r : C_r(K) \to C_{r1}(K)$ on the generators

$$d_r(\langle v_0, \ldots, v_r \rangle) = \sum_{i=0}^{r} (-1)^i \langle v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_r \rangle$$

and then extend linearly. [2 marks, bookwork]

The kernel of the boundary homomorphism d_r is called the r-cycle group and denoted by $Z_r(K)$, i.e. $Z_r(K) = \{c \in C_r(K) \mid d_r(c) = 0\}$. [1 mark, bookwork]

The image of the boundary homomorphism d_{r+1} is called the r-boundary group and is denoted by $B_r(K)$, i.e. $B_r(K) = \{d_{r+1}(c) \mid c \in C_{r+1}(K)\}$. [1 mark, bookwork]

(b) $Z_1(K) \cong \mathbb{Z}^3$ is generated by

$$\begin{aligned} z_1 &= \langle v_1, v_3\rangle + \langle v_3, v_2\rangle + \langle v_2, v_1\rangle \\ z_2 &= \langle v_3, v_5\rangle + \langle v_5, v_2\rangle + \langle v_2, v_3\rangle \\ z_3 &= \langle v_5, v_4\rangle + \langle v_4, v_2\rangle + \langle v_4, v_5\rangle \end{aligned}$$

$B_1(K) \cong \mathbb{Z}$ is generated by z_2. [2 marks, similar to question set]

We obtain

$$H_1(K) = Z_1(K)/B_1(K) = (\mathbb{Z}z_1 \oplus \mathbb{Z}z_2 \oplus \mathbb{Z}z_3)/\mathbb{Z}z_2 \cong \mathbb{Z}^2.$$

[2 marks, similar to question set]

[Total: 10 marks]

Feedback: The question was generally done well. The most common mistake in (a) was a missing reference to the orientation of a simplex. In part (b) I have often seen notation like $\mathbb{Z}^3/\mathbb{Z}$. Note, that this does not make sense, since $\mathbb{Z}$ is not a subgroup of $\mathbb{Z}^3$ (although there are many subgroups of $\mathbb{Z}^3$ being isomorphic to $\mathbb{Z}$ but the quotient will depend on the choice of such a subgroup).

A4.

(a) Consider the simplicial complex K consisting of the two 4-simplices $\langle 0, e_1, e_2, e_3, e_4\rangle \in \mathbb{R}^4$ and $\langle 0, e_1, e_2, e_3, -e_4\rangle \in \mathbb{R}^4$ (which intersect in a 3-simplex as a common face) and all their faces. Give an argument why the homology groups of K are given by

$$H_i(K) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & \text{else.} \end{cases}$$

(b) Let L be the 3-skeleton of K. Calculate the Euler characteristic of L and find its simplicial homology groups.

[10 marks]

Solution

(a) The underlying space is contractible. Indeed, $H(t, x) := t \cdot x$ gives a homotopy between the identity and the constant map. Because of the homotopy invariance of homology the homology groups are the same as for a point. [3 marks, similar to question set]

(b) For the Euler characteristic of $|K|$ we have $\chi(|K|) = \chi(*) = 1$, since the Euler characteristic depends only on the ranks of the homology groups. On the other hand we have

$$\chi(L) = \sum_{r=0}^{3} (-1)^r n_r = \chi(K) - n_4 = \chi(K) - 2 = -1,$$

since we have exactly two 4-dimensional simplices in K. [2 marks, similar to question set]

Now L is 3-dimensional and so and so has trivial homology groups in dimensions above 3. For $0 \leqslant i \leqslant 3$ we have $C_i(K) = C_i(L)$ and the boundary homomorphisms are the same. Hence, $H_i(K) = H_i(L)$ for $0 \leqslant i \leqslant 2$. Since $C_4(L) = 0$ we have $H_3(L) = Z_3(L)$ a free group of rank β_3. Now, using the identity $-1 = \chi(L) = \sum(-1)^i \beta_i$ we obtain $1 - \beta_3 = -1$(since $\beta_1 = \beta_2 = 0$) and so $\beta_3 = 2$ and $H_3(L) = \mathbb{Z}^2$. [5 marks, similar to question set]

Feedback: Most people used the correct approach to solve the problem. Occasionally people mixed up the notion of homotopy equivalence with that of homeomorphic spaces. Also some people seemed to be confused by the fact that an r-simplex has $r + 1$ vertices (when using binomial coefficients to count simplices)

SECTION B

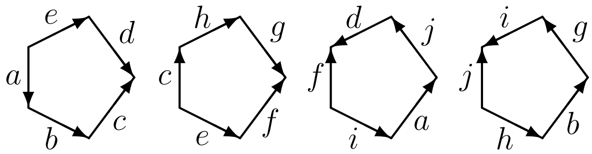
Answer **THREE** of the FOUR questions.

B5.

(a) Explain how a *surface symbol* may be used to represent a closed surface arising from the identification in pairs of the edges of a polygon.

(b) State the *classification theorem* for surface symbols.

(c) The boundaries of four discs are identified as shown below



Find a symbol for the resulting closed surface. By reducing the symbol to canonical form, or otherwise, identify the surface up to homeomorphism.

[15 marks]

Solution

(a) To write down a symbol respresenting a topological polygon with edges identified in pairs a letter is assigned to each edge of the polygon, assigning the same letter to two edges if and only if they are to identified. starting at any vertex, write down the letters in sequence going around the boundary, assigning the exponent -1 at the second appearance if the order to the vertices is reversed. [4 marks, bookwork]

(b) The classification theorem states that every closed surface is representatable by one and only one of the following symbols:

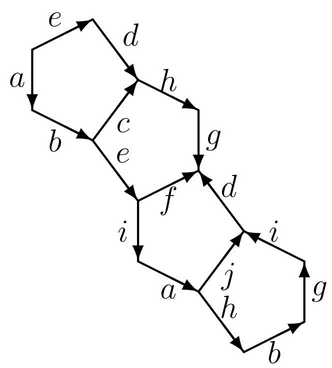
(i) xx^{-1},

(ii) $x_1y_1x_1^{-1}y_1^{-1}\dots x_gy_gx_g^{-1}y_g^{-1}$,

(iii) $x_1x_1\dots x_gx_g$.

[2 marks, bookwork]

(c) We can produce a single polygon with edges to be identified in pairs by using three edge identifications to join up the four polygons as follows.



This may be represented by the symbol $abeiahbgidg^{-1}h^{-1}d^{-1}e^{-1}$

[3 marks, similar to question set]

Now reducing the symbol for the polygon identifications to standard form gives the following.

$$
\begin{aligned}
\dot{a}bei\dot{a}hbgidg^{-1}h^{-1}d^{-1}e^{-1} &\sim (aa)i^{-1}\dot{e}^{-1}b^{-1}hbgidg^{-1}h^{-1}d^{-1}\dot{e}^{-1} \\
&\sim (aa)\ddot{i}^{-1}(e^{-1}e^{-1})dhgd^{-1}\ddot{i}^{-1}g^{-1}b^{-1}h^{-1}b \\
&\sim (aa)(e^{-1}e^{-1})(i^{-1}i^{-1})d\dot{g}^{-1}h^{-1}d^{-1}\dot{g}^{-1}b^{-1}h^{-1}b \\
&\sim (aa)(e^{-1}e^{-1})(i^{-1}i^{-1})d(g^{-1}g^{-1})dhb^{-1}h^{-1}b \\
&\sim (aa)(e^{-1}e^{-1})(i^{-1}i^{-1})(g^{-1}g^{-1})(dd)(hb^{-1}h^{-1}b) \\
&\sim (x_1x_1x_2x_2x_3x_3x_4x_4x_5x_5)(x_6y_6x_6^{-1}y_6^{-1}) \\
&\sim x_1x_1x_2x_2x_3x_3x_4x_4x_5x_5x_6x_6x_7x_7
\end{aligned}
$$

[5 marks, similar to question set]

Hence the surface is non-orientable of genus 7. [1 marks, similar to question set]

[**Total: 15 marks**]

Feedback: Most people did well here. There was some confusion on part (a). Occasionally people didn't remember the algorithm to reduce symbols.

B6. Let e_i be the ith standard basis vector in $\mathbb{R}^8$, $1 \leqslant i \leqslant 8$. Consider the set K of sixteen triangles with vertices e_i , e_j and e_k where ijk runs over the following triples:

$$126, 236, 138, 148, 348, 146, 365, 345, 467, 675, 472, 751, 452, 152, 237, 137.$$

(a) Verify that K is a simplicial surface, [For the link condition, you need only check the vertices e_1 and e_8 to illustrate the method.]

(b) Represent the underlying space of K as a polygon with edges identified in pairs.

(c) Calculate the Euler characteristic of K and determine whether it is orientable.

(d) Identify the underlying space of K up to homeomorphism.

[15 marks]

Solution

(a) The intersection condition is satisfied automatically since the vertices are linearly independent.

[1 mark]

The connectivity condition is satisfied because (for example) the following edges link all of the vertices: $e_8 - e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7$. [1 mark]

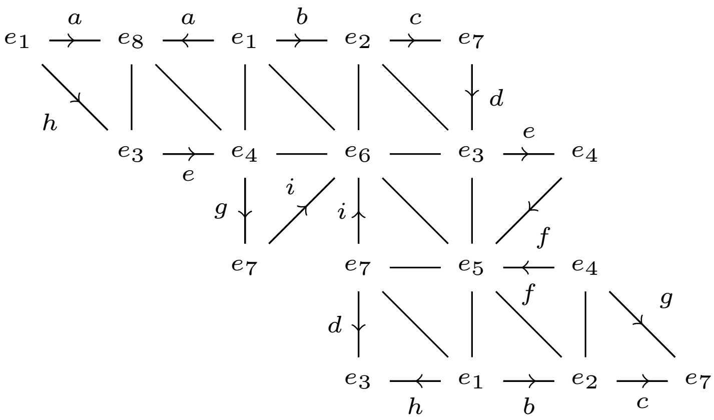
For the link condition look at

link(e_1): $e_7 - e_3 - e_8 - e_4 - e_6 - e_2 - e_5 - e_7$

link(e_8): $e_1 - e_3 - e_4 - e_1$

Which are closed polygons. [2 mark]

(b) A corresponding polygon with edges identified in pairs might look as follows.



[5 marks]

(c) For th Euler characteristic note, that we have 8 vertices and 16 triangles. For the number of edges we obtain $e(K) = \frac{3}{2} \cdot 16 = 24$. Hence, $\chi(K) = 8 - 24 + 16 = 0$. [2 marks]

K is orientable. This is best seen, when considering the corresponding polygon with pairwise identified edges. There, every edges occurs twice on the boundary, but in opposite directions. This implies orientability. Indeed, we may orientate all triangles, say clockwise, in the picture above. It's obvious that these orientations are coherent along the inner edges. A direct check shows coherence also along the boundary edges (this is exactly due to the fact that every edges occurs in clockwise and anti-clockwise direction along the boundary). [2 marks]

(d) By the Classification Theorem the underlying space of K is homeomorphic to the torus T_1.

[2 marks]

[Total: 15 marks, similar to question set]

Feedback: The most problems occured in part (c) when talking about orientability. Occasionally people forgot to make a statement at all. More often people correctly state that the surface is orientable but their argument was incomplete. It is important then when choosing to orient all triangles e.g. clockwise in the plane, that then the orientations are also coherent along the boundary edges.

B7.

(a) Outline the definition of the connected sum $S_1 \# S_2$ of two connected surfaces S_1 and S_2.

(b) State, and outline the proof of, the relationship between $\chi(S_1)$, $\chi(S_2)$ and $\chi(S_1 \# S_2)$.

(c) Calculate the Euler characteristic of the surfaces that arise in the classification theorem for compact connected surfaces. [You may assume the Euler characteristic of the 2-sphere, the torus and the projective plane without proof.]

(d) Explain the rôle of the Euler characteristic in proving the classification theorem for connected compact surfaces.

[15 marks]

Solution

(a) Suppose that S_1 and S_2 are non-empty path-connected topological surfaces. Choose subspaces $V_1 \subset S_1$ and $V_2 \subset S_2$ which are homeomorphic to the open disc $B_1(\mathbf{0}) \subset \mathbb{R}^2$ by homeomorphisms

$$\phi_i \colon B_1(\mathbf{0}) \to V_i \quad \text{for } i = 1 \text{ and } i = 2$$

We obtain the connected sum by removing the interiors of smaller discs, i.e. $\phi_i\big(B^2_{1/2}(\mathbf{0})\big)$ and glue along the boundary circles. More precisely, we define the quotient space of the disjoint union

$$S = \Big[\Big(S_1 - \phi_1\big(B^2_{1/2}(\mathbf{0})\big)\Big) \sqcup \Big(S_2 - \phi_2\big(B^2_{1/2}(\mathbf{0})\big)\Big)\Big] \Big/ \!\sim$$

where $\phi_1(\mathbf{u}) \sim \phi_2(\mathbf{u})$ for $\mathbf{u} \in B^2_1(\mathbf{0})$ with $|\mathbf{u}| = 1/2$.

[4 marks]

(b) We have the following relation $\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$.

Suppose that K_1 and K_2 are simplicial surfaces such that $|K_1| \cong S_1$ and $|K_2| \cong S_2$. Then we may construct simplicial surface K such that $|K| \cong S_1 \# S_2$ by removing a triangle from each of K_1 and K_2 and identifying the vertices and edges of this triangle. Then:
$f(K) = f(K_1) + f(K_2) - 2$,
$e(K) = e(K_1) + e(K_2) - 3$,
$v(K) = v(K_1) + v(K_2) - 3$.
Hence $\chi(K) = \chi(K_1) + \chi(K_2) - 2 + 3 - 3 = \chi(K_1) + \chi(K_2) - 2$ as required. [5 marks]

(c) By induction on g we obtain $\chi(T_g) = 2 - 2g$, since $\chi(T_1) = 0$ and

$$\chi(T_{g+1}) = \chi(T_g + T_1) = \chi(T_g) + \chi(T_1) - 2 = (2 - 2g) - 2 = 2 - 2(g + 1).$$

Similarly for P_g, since $\chi(P_1) = 1$ we obtain $\chi(P_g) = 2 - g$ by induction

$$\chi(P_{g+1}) = \chi(P_g + P_1) = \chi(P_g) + \chi(P_1) - 2 = (2 - g) + 1 - 2 = 2 - (g + 1).$$

[4 marks]

(d) The Euler characteristic is used in the proof of the classification theorem to help to distinguish the spaces in the list. Indeed, all orientable surfaces can be distinguished from each other by the Euler characteristic and all non-orientable surfaces from each other. However, $\chi(P_{2g}) = \chi(T_g)$. Hence, we need additionally the orientability property. [2 marks]

[Total: 15 marks]

Feedback: Most people did well, but it's import to note that even in an outline I want to see precise statement, i.e. only saying that we remove open discs and glue along their boundaries was not enough (and is even wrong if one chooses these discs not carefully).

B8.

(a) Let K and L be simplicial complexes. Define what is meant by a *simplicial map* $|K| \to |L|$ (with respect to K and L). Define what is meant by a *simplicial approximation* to a continuous map $f\colon |K| \to |L|$ (with respect to K and L).

(b) Prove that a simplicial approximation to f is homotopic to f.

(c) Let $K = L$ be the 1-skeleton of the 2-simplex $\bar{\Delta}^2$ so that $|K| \cong S^1(\subset \mathbb{C})$ by radial projection. Let $f\colon S^1 \to S^1$ be the function given by $f(z) = z^2$. Prove that the corresponding function $f\colon |K| \to |L|$ does not have a simplicial approxiation with respect to K and L but with respect to $|K'|$ and $|L|$, where K' denotes the barycentric subdivision of K.

[15 marks]

Solution

(a) A map of simplicial complexes $s : K \to L$ is induced by a map of the vertex sets $s_0 : V(K) \to V(L)$ so that if $\{v_0, v_1, \ldots, v_r\}$ is an r-simplex of K then $\{s_0(v_0), s_0(v_1), \ldots, s_0(v_r)\}$ is a simplex in L (possibly of lower dimension since s_0 need not be an injection on the vertices of the simplex). Such a map of the vertices may be extended by linearity over the simplices and gives a continuous function $|s| : |K| \to |L|$ by the Gluing Lemma. A function between the underlying spaces which arises in this way is called a simplicial map. [3 marks, bookwork]

We say that a simplicial map $|s| : |K| \to |L|$ is a simplicial approximation to a continuous map $f : |K| \to |L|$ if, for each point $x \in |K|$, the point $|s|(x)$ belongs to the carrier of $f(x)$ i.e. simplex of L whose interior contains $f(x)$. [2 marks, bookwork]

(b) Define $H\colon |K| \times I \to |L|$ by $H(x,t) = (1-t)|s|(x) + tf(x) \in |L|$. This definition makes sense since both of $|s|(x)$ and $f(x)$ lie in the carrier of $f(x)$ which is a convex subset of Euclidean space. As a composition of continuous functions H is clearly continuous. [2 marks, bookwork]

(c) Radial projection gives a homeomorphism $h\colon |K| \to S^1$, which is the identity on $v_1 = 1$, $v_2 = \exp(\frac{2}{3}\pi i)$ and $v_3 = \exp(\frac{4}{3}\pi i)$, since those point lie already on S^1. Then $f(z) = z^2$ corresponds to a function $g\colon |K| \to |K|$ with $g(v_1) = v_1$, $g(v_2) = v_3$ and $g(v_3) = v_2$. So a simplicial approximation $|\phi|$ to g would have to be given by this vertex map. Now, consider $x = (v_1 + v_2)/2 = h^{-1}(e^{\pi i/3})$, whose carrier is $\langle v_1, v_2\rangle$. We have then $g(x) = v_2$, but $|\phi|(x) = (v_1 + v_3)/2 \notin \mathrm{carr}_L(v_2) = v_2$.

[4 marks, bookwork]

The first barycentric subdivision introduces new vertices at $w_1 = h^{-1}\big(e^{\pi i/3}\big)$, $w_2 = h^{-1}(-1)$ and $w_3 = h^{-1}\big(\exp(5\pi i/3)\big)$. Then the simplicial map corresponding to the admissible vertex map $v_1 \mapsto v_1$, $w_1 \mapsto v_2$, $v_2 \mapsto v_3$, $w_2 \mapsto v_1$, $v_3 \mapsto v_2$ and $w_3 \mapsto v_3$ actually is the function g and so is certainly a simplicial approximation to it. [4 marks, bookwork]

[Total: 15 marks]

Feedback: Only a few people attempted the question.

SECTION C

Answer **ALL** THREE questions.

C9.

(a) Define what is meant by the *degree* $\deg(f)$ of a continuous selfmap $f\colon S^n \to S^n$ of a sphere. Show that your definition does not depend one the choice of a triangulation.

(b) Calculate the degree of the map $f\colon S^1 \to S^1$ given by $f(z) = z^2$ for $z \in S^1 \subset \mathbb{C}$.

(c) Given two continuous maps $f, g\colon S^n \to S^n$ show that $\deg(f \circ g) = \deg(f) \cdot \deg(g)$ holds.

(d) Show that a homeomorphism $f\colon S^n \to S^n$ has either degree 1 or -1.

[16 marks]

Solution

(a) Consider a triangulation $h\colon |K| \to S^n$ of the of the n-sphere. Then $f\colon S^n \to S^n$ induces an map $(h^{-1} \circ f \circ h)\colon |K| \to |K|$. Hence, a homomorphism $(h^{-1} \circ f \circ h)_*\colon H^n(K) \to H^n(K)$. Since $H^n(K) \cong \mathbb{Z}$ the homomorphism $(h^{-1} \circ f \circ h)_*$ must be given by $[z] \mapsto \lambda[z]$ for some $\lambda \in \mathbb{Z}$. We then define $\deg(f) = \lambda$. [4 mark, bookwork]

Assume, there is a second triangulation $k\colon |L| \to S^n$. Then we have a homeomorphism $g = h^{-1} \circ k\colon |L| \to |K|$. With this definition we obtain

$$(k^{-1} \circ f \circ k)_*([z]) = (g^{-1} \circ h^{-1} \circ f \circ h \circ g)_*([z]) = g_*^{-1}((h^{-1} \circ f \circ h)_*(g_*([z]))) = g_*^{-1}(\lambda g_*([z])) = \lambda[z],$$

Where the equalities follow from the definition of g, the functoriality of homology, the definition of λ above and the fact that g_*^{-1} is a homomorphism. [3 mark, bookwork]

(b) We consider the triangulation K and simplicial approximation g from B8 c). Note, that a generator of $H^n(K) = H^n(K')$ (which get identified via the isomorphism χ_*) is given by

$$z = [\langle v_1, w_1\rangle + \langle w_1, v_2\rangle + \langle v_2, w_2\rangle + \langle w_2, v_3\rangle + \langle v_3, w_3\rangle + \langle w_3, v_1\rangle]$$

(which gets identified with the homology class $[\langle v_1, v_2\rangle + \langle v_2, v_3\rangle + \langle v_3, v_1\rangle]$ via χ_*).

Now,

$$f_*([z]) = [\langle v_1, v_2\rangle + \langle v_2, v_3\rangle + \langle v_3, v_1\rangle + \langle v_1, v_2\rangle + \langle v_2, v_3\rangle + \langle v_3, v_1\rangle] = 2[z].$$

Hence, $\deg(f) = 2$.

[4 mark, bookwork]

(c) $\deg(f \circ g)$ is given by

$$(f \circ g)_*([z]) = f_*(g_*([z])) = f_*(\deg(g)[z]) = \deg(g) \cdot (f_*([z])) = \deg(g)\deg(f)[z].$$

I.e. $\deg(f \circ g) = \deg f \cdot \deg g$. Note, that the equalities above follow from the functoriality of homology, the definition of $\deg(g)$, the fact that f_* is a group homomorphism and the definition of $\deg(f)$. [3 mark, bookwork]

(d) Consider $\mathrm{id}_{S^n} = f \circ f^{-1}$. By the above and the fact that $\deg(\mathrm{id}) = 1$ one obtains $1 = \deg(f) \cdot \deg(f^{-1})$. But 1 and -1 are the only elements of $\mathbb{Z}$ with multiplicative inverses.

[2 mark, bookwork]

[**Total: 16 marks**]

Feedback: Those who attempted the question usually did well. However part (c) was often left out.

C10.

(a) Define what is meant by saying that (X, A) is a *triangulable pair* of spaces.

(b) State the axioms for the reduced homology groups of triangulable spaces.

(c) Prove, from the axioms, that a homotopy equivalence of triangulable spaces $f: X \to Y$ induces isomorphisms $f_*: \tilde{H}_k(X) \to \tilde{H}_k(Y)$ of reduced homology groups.

(d) Determine from the axioms the reduced homology groups of the wedge sum of two spheres $S^n \vee S^n$. [For this you may assume the reduced homology of the sphere without proof.]

[17 marks]

Solution

(a) A *triangulable pair* of spaces (X, A) is a topological space X with a subspace A such that there is a homeomorphism $h: X \to |K|$, the underlying space of a simplicial complex K, with $h(A) = |L|$ the underlying space of a subcomplex L of K. [2 mark, bookwork]

(b) A *reduced homology theory* assigns to each non-empty triangulable space X a sequence of abelian groups $\tilde{H}_n(X)$ for $n \in \mathbb{Z}$ and for each continuous map of triangulable spaces $f: X \to Y$ a sequence of homomorphisms $f_*: \tilde{H}_n(X) \to \tilde{H}_n(Y)$ such that the following axioms hold.

(i) [Functorial Axiom 1] Given continuous functions $f: X \to Y$ and $g: Y \to Z$, it follows that

$$g_* \circ f_* = (g \circ f)_*: \tilde{H}_n(X) \to \tilde{H}_n(Z) \text{ for all } i.$$

(ii) [Functorial Axiom 2] For the identity map $\mathrm{id}_X: X \to X$,

$$(\mathrm{id}_X)_* = \mathrm{id}_{H_n(X)}: \tilde{H}_n(X) \to \tilde{H}_n(X) \text{ (the identity homomorphism) for all } n.$$

(iii) [Homotopy Axiom] For homotopic maps $f \simeq g: X \to Y$,

$$f_* = g_*: \tilde{H}_n(X) \to \tilde{H}_n(Y) \text{ for all } n.$$

(iv) [Exactness Axiom] For any triangulable pair (X, A) there are boundary homomorphisms $\partial\colon \tilde{H}_n(X/A) \to \tilde{H}_{n-1}(A)$ for all n which fit into a long exact sequence as follows.

$$\ldots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{q_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \xrightarrow{i_*} \tilde{H}_{n-1}(X) \to \ldots$$

Furthermore, given any continuous function of triangulable pairs $f\colon (X, A) \to (Y, B)$ (i.e. $f\colon X \to Y$ such that $f(A) \subset B$) this induces a continuous function of quotient spaces $\bar{f}\colon X/A \to Y/B$. Then the following diagram commutes for all n.

$$\begin{array}{ccc} \tilde{H}_n(X/A) & \xrightarrow{\partial} & \tilde{H}_{n-1}(A) \\ \big\downarrow \bar{f}_* & & \big\downarrow f_* \\ \tilde{H}_n(Y/B) & \xrightarrow{\partial} & \tilde{H}_{n-1}(B) \end{array}$$

(v) [Dimension Axiom] $\tilde{H}_0(S^0) \cong \mathbb{Z}$ and $\tilde{H}_n(S^0) = 0$ for all $n \neq 0$.

[7 marks, bookwork]

(c) Suppose $f\colon X \to Y$ and $g\colon Y \to X$ are homotopy inverses to each other, i.e. $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. Then by (i) and (ii) we have

$$f_* \circ g_* = (f \circ g)_* = (\mathrm{id}_Y)_* = \mathrm{id}_{\tilde{H}(Y)}$$

and

$$g_* \circ f_* = (g \circ f)_* = (\mathrm{id}_X)_* = \mathrm{id}_{\tilde{H}(X)}\,.$$

Hence, f_* and g_* are inverse to each other and $\tilde{H}(X) \cong \tilde{H}(Y)$. [3 marks, problem set

(d) Consider $X_1 \cong S^n$ and $X_2 \cong S^n$ and the pair $(X_1 \vee X_2, X_1)$. One has $(X_1 \vee X_2)/X_1 \cong X_2$. Then the exactness axiom gives

$$\ldots \to \tilde{H}_{k+1}(X_2) \xrightarrow{\partial} \tilde{H}_k(X_1) \xrightarrow{i_*} \tilde{H}_k(X_1 \vee X_2) \xrightarrow{q_*} \tilde{H}_k(X_2) \xrightarrow{\partial} \tilde{H}_{k-1}(X_1) \to \ldots$$

For $k \neq n$ one has $\tilde{H}_k(X_2) = \tilde{H}_k(X_1) = 0$. Hence, $\tilde{H}_k(X_1 \vee X_2) = 0$. For $k = n$ we have $\tilde{H}_n(X_2) = \tilde{H}_n(X_1) = \mathbb{Z}$ and and $\tilde{H}_{n+1}(X_2) = \tilde{H}_{n-1}(X_1) = 0$. Hence, the is a short exact sequence

$$0 \to \mathbb{Z} \to \tilde{H}_n(X_1 \vee X_2) \to \mathbb{Z} \to 0,$$

which implies $\tilde{H}_n(X_1 \vee X_2) \cong \mathbb{Z} \times \mathbb{Z}$.

[5 marks, similar to problem set]

[**Total: 17 marks**]

Feedback: When stating the axioms people forgot to tell what $\tilde{H}_r(X)$ and f_* actually are. Only a few people attempted part (d). Often they put mor effort into it then it was actually needed by proving the statement for general X_1 and X_2.

C11.

(a) Define what is meant by a *finite cellular complex.*

(b) Define the cellular chain groups and the cellular boundary maps.

(c) Give a cellular decomposition of the projective plane.

(d) Calculate the cellular homology groups for the projective plane.

[17 marks]

Solution

(a) A finite cell complex (or finite CW complex) is a Hausdorff space X which is a finite disjoint union $X = \coprod_\alpha e_\alpha$ of subspaces that are open cells, together with a a characteristic map for every cell e_α: a continuous map $f_\alpha \colon \mathbb{D}^k \to X$ that maps B^k homeomorphically to e_α and maps the boundary $\partial B = S^{k-1}$ continuously into a union of cells e_β that all have smaller dimension than k. [4 marks, bookwork]

(b) Given a cell complex X we define the cellular chain groups as

$$C_k(X) = \tilde{H}_k(X^k/X^{k-1})$$

with *boundary maps*

$$d_k : C_k(X) \to C_{k-1}(X)$$

given by the composition $d_k = (p_{k-1})_* \circ \partial_k$. Where $\partial_k : \tilde{H}_k(X^k/X^{k-1}) \to \tilde{H}_{k-1}(X^{k-1})$ is the corresponding homomorphism from the long exact sequence for the pair (X^k, X^{k-1}) and $p_{k-1} : X^{k-1} \to X^{k-1}/X^{k-2}$ is the contraction map.

The cellular homology groups are defined by

$$H_k^{\text{cell}}(X) = \ker d_k / \operatorname{im} d_{k+1}.$$

[5 marks, bookwork]

(c) Using the construction $\mathbb{P}^2 \cong {}^{\mathbb{D}^2}/_{\sim}$ with $x \sim -x$ for points on the boundary we obtain a decomposition of $\mathbb{P}^2$ into one 2-cell e_2, one 1-cell e_1 and one 0-cell e_0. The 2-cell e_2 is given as the image of the interior under the quotient map $q \colon \mathbb{D}^2 \to \mathbb{P}^2$ and the characteristic map f_2 is just the quotient map itself. The one 1-cell e_1 coincides with the image of $S^1 \setminus \{(1,0), (-1,0)\}$. A characteristic map $f_1 \colon [-1,1] \to \mathbb{P}^2$ is given by $t \mapsto [e^{t \cdot \pi i}] \in {}^{S^1}/_{\sim} \subset {}^{\mathbb{D}^2}/_{\sim} = \mathbb{P}^2$. The unique 0-cell e_0 is the image of $(1,0)$ and $(-1,0)$ and comes with the obvious characteristic map f_0.

[4 marks, question set]

(d) Since the boundary of e_1 consists only of one point we have $d_1 \equiv 0$. For d_2 note that $X^1/X^0 = X^1$ consists just of ${}^{S^1}/_{\sim} \cong S^1$ and the map $\varphi_{2,1} \colon S^1 \to X^1/X^0 = S^1$ is just the quotient map $S^1 = \partial\mathbb{D}^2 \to {}^{S^1}/_{\sim} \cong S^1$ (which is given by $z \mapsto z^2$). We have seen, that this is a degree-2 map. Hence, we obtain

$$H_k^{\text{cell}}(X) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/2\mathbb{Z} & k = 1 \\ 0 & \text{else.} \end{cases}$$

[4 marks, bookwork]

[**Total: 17 marks**]

Feedback: Only a few people attemted the question. But those who tried usually did well. Often the characteristic maps where missing in the description of the cell decomposition of the projective plane. Also just drawing pictures of the k-cells was not sufficient.