MATH41071/MATH61071 *Algebraic topology*

**Solutions 6**

1. Using the simplicial complex $K$ of Solutions 4, Question 3(b) the subcomplex $L$ whose underlying space is the boundary circle of the Möbius band consists of the edges $\langle v_1, v_3 \rangle$, $\langle v_3, v_4 \rangle$, $\langle v_2, v_4 \rangle$, $\langle v_2, v_5 \rangle$, $\langle v_5, v_6 \rangle$, $\langle v_1, v_6 \rangle$ and their vertices. Then $H_1(L) = Z_1(L) \cong \mathbb{Z}$ generated by 

   \[ x = \langle v_1, v_3 \rangle + \langle v_3, v_4 \rangle - \langle v_2, v_4 \rangle + \langle v_2, v_5 \rangle + \langle v_5, v_6 \rangle - \langle v_1, v_6 \rangle. \]

   But then, using the notation of Solutions 5, Question 3(v), $i_*(x) = x_1 + x_2$ and so the induced map in homology is given by $i_*(x) = [x_1 + x_2] = [x_1] + [x_2] = 2[x_1]$.

2. There are three conditions for an equivalence relation.

   **reflexivity:** Given a continuous function $f : X \to Y$ then $f \simeq f$. A homotopy is given by $H(x, t) = f(x)$.

   **symmetry:** Given homotopic functions $f_0 \simeq f_1 : X \to Y$ then $f_1 \simeq f_0$. Given a homotopy $H : f_0 \simeq f_1$ then a homotopy $K : f_1 \simeq f_0$ is given by $K(x, t) = H(x, 1 - t)$.

   **transitivity:** Given homotopic functions $f_0 \simeq f_1 : X \to Y$ and $f_1 \simeq f_2 : X \to Y$ then $f_0 \simeq f_2 : X \to Y$. Given homotopies $H : f_0 \simeq f_1$ and $K : f_1 \simeq f_2$ then a homotopy $L : f_0 \simeq f_2$ is given by

   \[
   L(x, t) = \begin{cases} 
   H(x, 2t) & \text{for } 0 \leq t \leq 1/2, \\
   K(x, 2t - 1) & \text{for } 1/2 \leq t \leq 1.
   \end{cases}
   \]

   This is well-defined since $H(x, 1) = f_1(x) = K(x, 0)$ and is continuous by the Gluing Lemma.

   Hence homotopy is an equivalence relation.

3. (a) Suppose that $X = \{a\}$, a one-point space $P = \{a\}$. Let $c : X \to P$ be the constant map $c(x) = a$ and $f : P \to X$ be an map $f(a) = x_0$ giving a homotopy equivalence. Then $c \circ f = I : P \to P$ and, since $f \circ c \simeq I : X \to X$ there is a homotopy $H : X \times I \to X$ such that $H(x, 0) = (f \circ c)(x) = x_0$ and $H(x, 1) = x$. Then for each $x \in X$ we may define a path from $x_0$ to $x$ in $X$ by $\gamma(t) = H(x, t)$.

   Hence $X$ is path-connected.

   (b) Given the notation of (a), the singleton subset $\{x_0\}$ is a deformation retract of $X$ since $H : i \circ r \simeq I : X \to X$ where $i$ is the inclusion map and $r : X \to \{x_0\}$ is the constant map.
Now, for any other point \( x_1 \in X \), we may define a homotopy \( K: i_1 \circ r_1 \simeq I \colon X \to X \) (where \( i_1 : \{x_1\} \to X \) is the inclusion map and \( r_1 : X \to \{x_1\} \) is the constant map) by

\[
K(x, t) = \begin{cases} 
H(x_1, 1 - 2t) & \text{for } 0 \leq t \leq 1/2, \\
H(x, 2t - 1) & \text{for } 1/2 \leq t \leq 1.
\end{cases}
\]

This map is well defined since, for \( t = 1/2 \), \( H(x_1, 0) = x_0 = H(x, 0) \). It is continuous by the Gluing Lemma.

4. (a) Any simplicial approximation of \( f \) must map \( 0 \mapsto 0 \) and \( 1 \mapsto 1 \). The only admissible vertex map \( V(K) \to V(L) \) which does this maps \( \frac{1}{2} \mapsto \frac{2}{3} \), which does not give a simplicial approximation to \( f \) since \( |\phi(\frac{2}{3}) = \frac{3}{4} \) does not lie in the carrier of \( f(\frac{1}{2}) = \frac{1}{3} \) which is \( [0, \frac{2}{3}] \).

(ii) If we look at simplicial maps \( |K'| \to |L| \) (using the first barycentric subdivision of \( K \) with vertices at \( 0, \frac{1}{3}, \frac{2}{3}, \frac{3}{3} \) and 1, we are still out of luck. Since \( 1 \mapsto 1 \) we must have \( \frac{2}{3} \mapsto \frac{2}{3} \) or \( \frac{3}{3} \mapsto 1 \). In the first case, points in the range \( \frac{2}{3} < x < \sqrt{2} \) fail the approximation condition and in the second case points in the range \( \frac{1}{3} < x < \frac{2}{3} \) fail the condition.

(iii) However, using the second barycentric subdivision of \( K \), there is a simplicial map \( K'' \to L \) giving a simplicial approximation. For example, send all the vertices of \( K'' \) to 0 apart from \( \frac{2}{3} \mapsto \frac{2}{3} \) and \( 1 \mapsto 1 \).

(b) We have to take \( K^{(2)} \) consisting of the intervals \( [0, \frac{1}{4}], [\frac{1}{4}, \frac{1}{2}], [\frac{1}{2}, \frac{3}{4}] \) and \( [\frac{3}{4}, 1] \) and their endpoints. Now, one observes that

\[
\begin{align*}
\text{star}(0) &= \{0, \frac{1}{4}\} \subset [0, \frac{1}{4}) = f^{-1}(\text{star}(v_5)), \\
\text{star}(\frac{1}{4}) &= \{0, \frac{1}{2}\} \subset [0, \frac{1}{2}) = f^{-1}(\text{star}(v_6)), \\
\text{star}(\frac{1}{2}) &= \{\frac{1}{4}, \frac{3}{4}\} \subset [0, \frac{3}{4}) = f^{-1}(\text{star}(v_5)), \\
\text{star}(\frac{3}{4}) &= \{\frac{1}{2}, 1\} \subset (\frac{1}{2}, 1] = f^{-1}(\text{star}(v_2)), \\
\text{star}(1) &= \{\frac{3}{4}, 1\} \subset (\frac{1}{2}, 1] = f^{-1}(\text{star}(v_2)).
\end{align*}
\]

Hence, by (a) the vertex map \( s \) given by \( s(0) = s(\frac{1}{4}) = s(\frac{1}{2}) = v_5 \) and \( s(\frac{3}{4}) = s(1) = v_2 \) defines a simplicial approximation to \( f \).

5. Radial projection gives a homeomorphism \( h \colon |K| \to S^1 \). Set \( v_0 = 1 \), \( v_1 = e^{\frac{2}{3}\pi i} \) and \( v_2 = e^{\frac{4}{3}\pi i} \). Then \( f(z) = z^2 \) corresponds to a function \( g \colon |K| \to |K| \) with \( g(v_0) = v_0 \), \( g(v_1) = v_2 \) and \( g(v_2) = v_1 \). So a simplicial approximation to \( g \) would have to be given by this vertex map. However, the simplicial map
coming from this admissible vertex map is $z \mapsto \bar{z} = z^{-1}$ which is not homotopic to $z \mapsto z^2$. You can easily find points where the simplicial approximation condition fails, e.g. $g(h^{-1}(-1)) = h(f(-1)) = 1$ whose carrier is $\langle v_0 \rangle$, but $g(h^{-1}(-1)) = h(-1) \notin \langle v_0 \rangle$.

The first barycentric subdivision introduces new vertices at $w_0 = h^{-1}(e^{\pi i/3})$, $w_1 = h^{-1}(-1)$ and $w_2 = h^{-1}(e^{5\pi i/3})$. Then the simplicial map corresponding to the admissible vertex map $w_1 \mapsto v_1, v_1 \mapsto v_2, w_1 \mapsto v_0, v_2 \mapsto v_1, w_2 \mapsto v_2$ and $v_0 \mapsto v_0$ actually is the function $g$ and so is certainly a simplicial approximation to it.

6. First of all notice that if a function $f : S^m \to S^n$ is not a surjection then it is homotopic to a constant function since, if $v \in S^n$ then $S^n \setminus \{v\} \cong \mathbb{R}^n$. This means that if $v$ is not a value of $f$ then $f$ factors as $\phi^{-1} \circ f_1$ where $\phi : S^n \setminus \{v\} \to \mathbb{R}^n$ is a homeomorphism and $f_1 = \phi \circ f : S^m \to \mathbb{R}^n$. Then the homotopy $H : S^m \times I \to S^n$ defined by $H(x,t) = \phi^{-1}(t f_1(x))$ shows that $f$ is homotopic to a constant function.

A homeomorphism $\phi : S^n \setminus \{v\} \to \mathbb{R}^n$ is given by stereographic projection from $v$: we map a point $x \in S^n \setminus \{v\}$ to the point where the line through $v$ and $x$ meets the hyperplane $v^\perp$. Since a general point on this line is given by $x + t(v-x)$ we find the point where this cuts the line by solving $(x + t(v-x)) \perp v = 0$ which gives $t = (x,v)(v-x)/(1-x,v).

At first sight this might appear to prove the required result since it seems obvious that, if $m < n$, the a continuous function $f : S^m \to S^n$ cannot be a surjection. However, this ‘obvious’ result is in fact false and remarkably you can find continuous surjections $S^m \to S^n$. I don’t know a good reference for this general result. However, by using a ‘space filling curve’ (a continuous surjection $I \to I^2$) you can construct a continuous surjection $S^1 \to S^2$ which demonstrates that the ‘obvious’ result is false in this case.

We can overcome this problem by use of the Simplicial Approximation Theorem. Let $K = (\Delta^{m+1})[m]$ and $L = (\Delta^{n+1})[n]$ so that $|K| \cong S^m$ and $|L| \cong S^n$. Then a continuous function $f : S^m \to S^n$ gives a continuous function $g : |K| \to |L|$ (by composing $f$ with homeomorphisms). By the Simplicial Approximation Theorem, $g$ is homotopic to a simplicial map $|K^{(r)}| \to |L|$ which is not a surjection since $|K^{(r)}|$ will be mapped to the underlying space of the $m$-skeleton of $L$. Hence $f$ is homotopic to a function $S^m \to S^n$ which is not a surjection.