<u>Two hours</u>

THE UNIVERSITY OF MANCHESTER

INTRODUCTION TO TOPOLOGY

14 January 2014

14:00 — 16:00

Answer **ALL** FOUR questions in Section A (40 marks in total). Answer **THREE** of the FOUR questions in Section B (45 marks in total). If more than THREE questions from Section B are attempted then credit will be given for the best THREE answers.

Electronic calculators are permitted, provided they cannot store text.

SECTION A

Answer **ALL** FOUR questions.

A1. (a) Define what is meant by a *topology* on a set X.

(b) Define what is meant by saying that a function $f\colon X \to Y$ between topological spaces is *continuous*. Define what is meant by saying that f is a *homeomorphism*.

(c) Prove that the punctured disc $X = \{ x \in \mathbb{R}^2 \mid 0 < |x| < 1 \}$ with the usual topology is homeomorphic to the cylinder $Y = S^1 \times (0,2) \subset \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$ with the usual topology.

[Here S^1 denotes the unit circle $\{ x \in \mathbb{R}^2 \mid |x| = 1 \}$.]

[10 marks]

A2. (a) Suppose that $q\colon X \to Y$ is a surjection between topological spaces. What is meant by saying that the topology on Y is the *quotient topology* determined by q?

(b) Define an equivalence relation on $I^2 \subset \mathbb{R}^2$ (with the usual topology) by $(x,0) \sim (x,1)$, $(0,y) \sim (1,y)$ and $(x,y) \sim (x,y)$ where $x,\ y \in I$. Let $I^2/{\sim}$ be the set of equivalence classes with the quotient topology. Using the universal properties of the quotient topology and of the product topology, prove that this topological space is homeomorphic to the torus $S^1 \times S^1$.

[Here I is the closed interval $[0,1]$. Any general theorems used in your proof should be clearly stated.]

[10 marks]

A3. Suppose that X is a topological space.

(a) What is meant by saying that two paths in X from $x \in X$ to $x' \in X$ are *homotopic*?

(b) Define the *product* $\sigma * \tau$ of two paths σ and τ in X, giving the condition for this to exist.

(c) Define $\pi_1(X, x_0)$, the *fundamental group* of X based at $x_0 \in X$.

[You should define the group product and state what needs to be proved in order to show that the group product is well-defined. You need not confirm that this product gives a group structure.]

[10 marks]

A4. (a) Explain how a continuous function $f\colon X \to Y$ induces a homomorphism $f_*\colon \pi_1(X,x_0) \to \pi_1\big(Y, f(x_0)\big)$. You should indicate why f_* is well-defined and why it is a homomorphism.

(b) Prove that the fundamental group is a topological invariant by proving that a homeomorphism $f\colon X \to Y$ induces an isomorphism $f_*\colon \pi_1(X,x_0) \to \pi_1\big(Y, f(x_0)\big)$.

[You may find it useful to state and prove the functorial properties of the fundamental group.]

[10 marks]

SECTION B

Answer **THREE** of the FOUR questions.

If more than THREE questions are attempted then credit will be given for the best THREE answers.

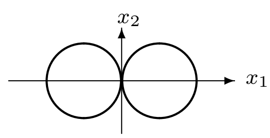
B5. (a) Define what is meant by saying that a topological space is *path-connected*.

(b) Prove that if $f\colon X \to Y$ is a continuous surjection of topological spaces and X is path-connected then Y is path-connected

(c) Define the *path-components* of a topological space X. [You need not verify that the equivalence relation used in the definition is an equivalence relation.]

(d) Let X be the subset of $\mathbb{R}^2$ with the usual topology defined by

$$X = \{\, \mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x} - (-1, 0)| = 1 \text{ or } |\mathbf{x} - (1, 0)| = 1 \,\}.$$



What are the path-components of $X \setminus \{(0, 0)\}$? Justify your answer.

[15 marks]

B6. (a) Define what is meant by saying that a topological space is *Hausdorff*.

(b) Determine whether the set $S = \{a, b, c\}$ with topology $\tau = \big\{\emptyset, S, \{a\}, \{b, c\}\big\}$ is Hausdorff.

(c) Suppose that X_1 is a subset of a topological space X. Define the *subspace topology* on X_1 induced by the topology on X. [It is not necessary to prove that this is a topology.]

(d) Prove that, if X is a Hausdorff space, then a subset X_1 of X with the subspace topology is also Hausdorff.

(e) Prove that, if a topological space X is Hausdorff, then all singleton subsets $\{a\}$ of X (where $a \in X$) are closed. Give an example to show that the converse of this statement is false; that is, if a topological space has all singleton subsets closed, it is not necessarily Hausdorff.

[15 marks]

B7. (a) Prove that, if the product $\sigma_0 * \tau_0$ of two paths σ_0 and τ_0 in a topological space X is defined and the paths σ_1 and τ_1 are homotopic to σ_0 and τ_0 respectively, then the product $\sigma_1 * \tau_1$ is defined and is homotopic to $\sigma_0 * \tau_0$.

(b) Prove that, if σ, τ and ρ are three paths in X such that the products $\sigma * \tau$ and $\tau * \rho$ are defined, then $(\sigma * \tau) * \rho$ and $\sigma * (\tau * \rho)$ are homotopic paths.

(c) Prove that, for topological spaces X and Y with points $x_0 \in X$, $y_0 \in Y$, there is an isomorphism of groups

$$\pi_1\big(X \times Y, (x_0, y_0)\big) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

[15 marks]

B8. (a) Let S^1 denote the unit circle in the complex plane with the usual topology. Explain how the continuous map $p\colon \mathbb{R} \to S^1$ given by $p(x) = \exp(2\pi i x)$ may be used to define the *degree* of a loop in S^1 based at 1.

(b) Explain how the degree may be used to define a group homomorphism $\phi\colon \pi_1(S^1, 1) \to \mathbb{Z}$ to the additive group of the integers.

(c) Find the degree of the loop $\sigma(s) = \exp(2\pi i n s)$ for an integer n, and hence prove that ϕ is an epimorphism.

(d) Prove that ϕ is a monomorphism.

[Theorems about the lifting of paths in S^1 to paths in $\mathbb{R}$ may be used without proof.]

[15 marks]

END OF EXAMINATION PAPER