Two hours

THE UNIVERSITY OF MANCHESTER

INTRODUCTION TO TOPOLOGY

20 January 2015

14:00 — 16:00

Answer **ALL** FOUR questions in Section A (40 marks in total). Answer **THREE** of the FOUR questions in Section B (45 marks in total). If more than THREE questions from Section B are attempted then credit will be given for the best THREE answers.

Electronic calculators are permitted, provided they cannot store text.

SECTION A

Answer **ALL** FOUR questions.

A1. (a) Define what is meant by a *topology* on a set X.

(b) Define what is meant by saying that a function $f\colon X \to Y$ between topological spaces is *continuous*. Define what is meant by saying that f is a *homeomorphism*.

(c) Prove that the annulus $X = \{\, \mathbf{x} \in \mathbb{R}^2 \mid 1 \leqslant |\mathbf{x}| \leqslant 2 \,\}$ with the usual topology is homeomorphic to the cylinder $S^1 \times [0,1] \subset \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$ with the usual topology.

[Here S^1 denotes the unit circle $\{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| = 1\,\}$.]

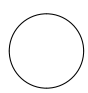
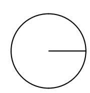
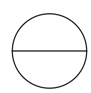
[10 marks]

A2. (a) Suppose that X is a topological space and x_0, x_1 are points of X. Define what is meant by a *path* in X from x_0 to x_1.

(b) Outline the definition of the *path-components* of a topological space X.

(c) Define what is meant by a *cut-point of type n* in a topological space X. What is meant by a *cut-pair of type n*?

(d) Explain how cut-points and cut-pairs may be used to prove that no two of the following subsets of the plane with the usual topology are homeomorphic.

(i)  (ii)  (iii) 

[The first set is a circle and each of the other two sets is the union of a circle and a line interval. You should assume that the end points of the line interval are included in the subset.]

[10 marks]

A3. (a) Define what is meant by saying that a topological space is *Hausdorff*.

(b) Determine whether the set $S = \{a, b, c\}$ with topology $\tau = \big\{\, \emptyset, S, \{a, c\}, \{b\} \,\big\}$ is Hausdorff.

(c) Suppose that X and Y are topological spaces. Define the *product topology* on the Cartesian product $X \times Y$. [It is not necessary to prove that this is a topology.]

(d) Prove that, if X and Y are Hausdorff spaces, then so is the product space $X \times Y$.

[10 marks]

A4. (a) Suppose that X is a topological space and x_0, x_1 are points of X. What is meant by saying that two paths in X from x_0 to x_1 are *homotopic*?

(b) Give a condition for the existence of the *product* $\sigma * \tau$ of two paths σ and τ in X and define the product.

(c) Prove that, if σ, τ and ρ are three paths in X such that the products $\sigma * \tau$ and $\tau * \rho$ exist, then $(\sigma * \tau) * \rho$ and $\sigma * (\tau * \rho)$ are homotopic paths.

[10 marks]

SECTION B

Answer **THREE** of the FOUR questions.

If more than THREE questions are attempted then credit will be given for the best THREE answers.

B5. (a) Suppose that $\sim$ denotes an equivalence relation on a topological space X. Define the *quotient topology* on the identification space $X/\!\sim$. State the *universal property* of the quotient topology.

(b) Suppose that $f\colon X \to Y$ is a continuous surjection from a compact topological space X to a Hausdorff topological space Y. Define an equivalence relation $\sim$ on X so that f induces a bijection $F\colon X/\!\sim\, \to Y$ from the identification space of this equivalence relation to Y. Prove that F is a homeomorphism. [State clearly any general results which you use.]

(c) Prove that the quotient space $(S^1 \times [0,1])/(S^1 \times \{1\})$ is homeomorphic to the closed unit disc D^2 (where S^1 and D^2 have the usual topology).

[15 marks]

B6. (a) Define what is meant by a *compact* subset K of a topological space X. Define what is meant by a *compact* topological space.

(b) Prove that, if $f\colon X \to Y$ is a continuous function of topological spaces and $K \subset X$ is a compact subset, then $f(K)$ is a compact subset of Y.

(c) Prove that, given closed non-empty subsets A_n, for $n \geqslant 1$, of a compact topological space X such that

$$A_1 \supset A_2 \supset \cdots \supset A_n \supset A_{n+1} \supset \cdots ,$$

then the intersection $\bigcap_{n=1}^{\infty} A_n$ is non-empty.

[Hint. Give a proof by contradiction.]

[15 marks]

B7. Suppose that X is topological space and that x_0 and x_1 are points of X.

(a) Define $\pi_1(X, x_0)$, the *fundamental group* of X based at x_0. You should define the group product and indicate why this is well-defined and gives a group structure.

(b) Let ρ be a path in X from x_0 to x_1. Explain how ρ may be used to define an isomorphism of fundamental groups:

$$u_\rho\colon \pi_1(X, x_0) \to \pi_1(X, x_1).$$

(c) Prove that, if X is path-connected and all paths ρ from x_0 to x_1 induce the same isomorphism, then $\pi_1(X, x_0)$ is abelian.

[Basic properties of the product of paths may be stated without proof.]

[15 marks]

B8. (a) Suppose that X_1 is a subspace of a topological space X. Define what is meant by saying that X_1 is a *retract* of X.

(b) Prove that the unit circle $S^1 = \{\, \mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| = 1 \,\}$ is a retract of the punctured plane $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ with the usual topology.

(c) Explain how a continuous function of topological spaces $f\colon X \to Y$ induces a homomorphism of fundamental groups $f_*\colon \pi_1(X, x_0) \to \pi_1\big(Y, f(x_0)\big)$ for $x_0 \in X$. You should indicate why f_* is a well-defined homomorphism.

(d) Use the functorial properties of the fundamental group to prove that, if X_1 is a retract of X, then, for any $x_0 \in X_1$, the homomorphism induced by the inclusion map

$$i_*\colon \pi_1(X_1, x_0) \to \pi_1(X, x_0)$$

is a monomorphism.

(e) Hence prove that S^1 is not a retract of $\mathbb{R}^2$.

[You may quote any fundamental groups that you need, without proof.]

[15 marks]

END OF EXAMINATION PAPER