Two hours

THE UNIVERSITY OF MANCHESTER

INTRODUCTION TO TOPOLOGY

26 January 2016

09:45 – 11:45

Answer **ALL** FOUR questions in Section A (40 marks in total). Answer **THREE** of the FOUR questions in Section B (45 marks in total). If more than THREE questions from Section B are attempted then credit will be given for the best THREE answers.

Electronic calculators are permitted, provided they cannot store text.

SECTION A

Answer **ALL** FOUR questions.

A1. (a) Define what is meant by a *topology* on a set X.

(b) Define what is meant by saying that a function $f\colon X \to Y$ between topological spaces is *continuous*. Define what is meant by saying that f is a *homeomorphism*.

(c) Prove that the punctured disc $\{\, x \in \mathbb{R}^2 \mid 0 < |x| < 1 \,\}$ with the usual topology is homeomorphic to the cylinder $S^1 \times (1,2) \subset \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$ with the usual topology.

[Here S^1 denotes the unit circle $\{x \in \mathbb{R}^2 \mid |x| = 1\,\}$.]

[10 marks]

A2. (a) Define what is meant by saying that a topological space X is *path-connected*.

(b) Define what is meant by saying that path-connectedness is a *topological property*?

(c) Prove that path-connectedness is a topological property.

(d) Prove that the unit circle $S^1 = \{\, x \in \mathbb{R}^2 \mid |x| = 1 \,\}$ with the usual topology is path-connected.

[10 marks]

A3. (a) Define what is meant by saying that a topological space is *Hausdorff*.

(b) Determine whether the set $S = \{a, b, c\}$ with topology $\tau = \big\{\emptyset, S, \{a\}, \{b, c\}\big\}$ is Hausdorff.

(c) Suppose that X_1 is a subset of a topological space X. Define the *subspace topology* on X_1 induced by the topology on X. [It is not necessary to prove that this is a topology.]

(d) Prove that, if X is a Hausdorff space, then a subset X_1 of X with the subspace topology is also Hausdorff.

[10 marks]

A4. (a) Suppose that X is a topological space and x_0, x_1 are points of X. What is meant by saying that two paths in X from x_0 to x_1 are *homotopic*?

(b) Define the *product* $\sigma * \tau$ of two paths σ and τ in X, giving the condition for the product to exist.

(c) Prove that, if the product $\sigma_0 * \tau_0$ of two paths σ_0 and τ_0 in X exists and the paths σ_1 and τ_1 are homotopic to σ_0 and τ_0 respectively, then $\sigma_1 * \tau_1$ also exists and is homotopic to $\sigma_0 * \tau_0$.

[10 marks]

SECTION B

Answer **THREE** of the FOUR questions.

If more than THREE questions are attempted then credit will be given for the best THREE answers.

B5. (a) Define what is meant by the *path-components* of a topological space.

[You may assume the definition of a path and properties of paths.]

(b) Prove that a continuous map of topological spaces $f\colon X \to Y$ induces a function

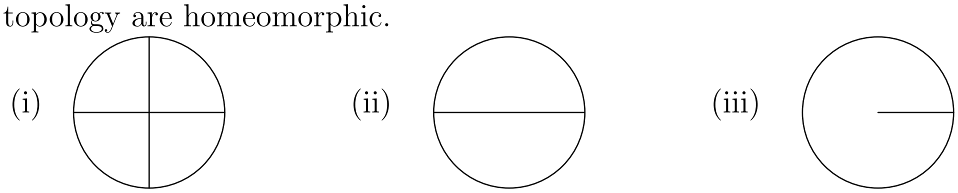
$$f_*\colon \pi_0(X) \to \pi_0(Y)$$

between the sets of path-components, taking care to prove that your function is well-defined.

Prove that if f is a homeomorphism then f_* is a bijection.

(c) A pair of distinct points $\{p, q\}$ in a topological space X is called a *cut-pair of type* n when the subspace $X \setminus \{p, q\}$ has n path-components. Prove that a homeomorphism $f\colon X \to Y$ induces a bijection between the subsets of cut-pairs of type n.

(d) Hence show, using cut-pairs of type 3 or otherwise, that no two of the following subspaces of $\mathbb{R}^2$ with the usual topology are homeomorphic.



[These diagrams represent a circle with two diameters, a circle with a single diameter and a circle with a single radial line.]

[15 marks]

B6. (a) Suppose that $q\colon X \to Y$ is a surjection from a topological space X to a set Y. Define the *quotient topology* on Y determined by q. State the *universal property* of the quotient topology.

(b) Suppose that $f\colon X \to Y$ is a continuous surjection from a compact topological space X to a Hausdorff topological space Y. Define an equivalence relation $\sim$ on X so that f induces a bijection $F\colon X/{\sim} \to Y$ from the identification space of this equivalence relation to Y. Prove that F is a homeomorphism.

[State clearly any general results which you use.]

(c) Let $X = \{\, x \in \mathbb{R}^2 \mid 1 \leqslant |x| \leqslant 2 \,\}$ with the usual topology. Prove that the identification space X/S^1 is homeomorphic to the closed unit disc $D^2 = \{\, x \in \mathbb{R}^2 \mid |x| \leqslant 1 \,\}$ with the usual topology.

[Here S^1 denotes the unit circle $\{x \in \mathbb{R}^2 \mid |x| = 1\,\}$.]

[15 marks]

B7. Suppose that X is a topological space and x_0 and x_1 are points of X.

(a) Define $\pi_1(X, x_0)$, the *fundamental group* of X based at x_0. You should define the group product and indicate why this is well-defined and gives a group structure.

(b) Define what is meant by saying that X is *simply connected*.

(c) Suppose that X is a path-connected space such that all paths from x_0 to x_1 are homotopic. Prove that X is simply connected.

[15 marks]

B8. (a) Let S^1 denote the unit circle in the complex plane with the usual topology. Explain how the continuous map $p\colon \mathbb{R} \to S^1$ given by $p(x) = \exp(2\pi i x)$ may be used to define the *degree* of a loop in S^1 based at 1.

(b) Explain how the degree may be used to define a group homomorphism $\phi\colon \pi_1(S^1, 1) \to \mathbb{Z}$ to the additive group of the integers. Prove that ϕ is an epimorphism.

[The homomorphism ϕ is in fact an isomorphism but you need not prove this.]

(c) Let $f\colon S^1 \to S^1$ be the map given by $f(z) = \overline{z}$ (the complex conjugate). Determine the homomorphism $g\colon \mathbb{Z} \to \mathbb{Z}$ corresponding to the induced homomorphism $f_*\colon \pi_1(S^1, 1) \to \pi_1(S^1, 1)$ using the isomorphism ϕ.

[Theorems about the lifting of paths in S^1 to paths in $\mathbb{R}$ may be used without proof.]

[15 marks]

END OF EXAMINATION PAPER