Two hours

THE UNIVERSITY OF MANCHESTER

TOPOLOGY

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Answer **ALL** FOUR questions in Section A (40 marks in total). Answer **THREE** of the FOUR questions in Section B (45 marks in total).

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL** FOUR questions.

A1.

(a) Define what is meant by a *topology* on a set X.

(b) Define what is meant by saying that a function $f\colon X \to Y$ between topological spaces is *continuous*. Define what is meant by saying that f is a *homeomorphism*.

(c) Prove that the closed unit disc $D^2 = \{x \in \mathbb{R}^2 \mid |x| \leqslant 1\}$ with the usual topology is homeomorphic to the hemisphere $\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\}$.

[Here S^2 denotes the unit sphere $\{x \in \mathbb{R}^3 \mid |x| = 1\}$ with the usual topology.]

A2.

(a) Define what is meant by saying that a topological space X is *path-connected*.

(b) What is meant by saying the path-connectedness is a *topological property*?

(c) Prove that path-connectedness is a topological property.

(d) Prove that

$$\{x \in \mathbb{R}^2 \mid |x - (0,1)| \leqslant 1 \text{ or } |x + (0,1)| \leqslant 1\} \subset \mathbb{R}^2$$

(with the usual topology) is path-connected.

A3.

(a) Define what is meant by saying that a topological space is *Hausdorff*.

(b) Determine whether the set $S = \{a, b, c\}$ with topology $\tau = \{\emptyset, \{a, c\}, \{b\}, \{a, b, c\}\}$ is Hausdorff.

(c) Suppose that X and Y are topological spaces. Define the *product topology* on the Cartesian product $X \times Y$. [It is not necessary to prove that this is a topology.]

(d) Prove that if $\Delta \subset X \times X$ is closed in the product topology, then X is Hausdorff.

A4.

(a) Suppose that X_1 is a subspace of a topological space X. Define what is meant by saying that X_1 is a *retract* of X.

(b) Prove that the unit circle $S^1 = \{x \in \mathbb{R}^2 \mid |x| = 1\}$ is a retract of the punctured disc $\mathbb{D}^2\ \{0\}$ where $\mathbb{D}^2 = \{x \in \mathbb{R}^2 || x| = 1\}$ with the usual topology.

(c) Use the functorial properties of the fundamental group to prove that, if X_1 is a retract of X, then, for any $x_0 \in X_1$, the homomorphism induced by the inclusion map

$$i \colon \pi_1(X_1, x_0) \to \pi_1(X, x_0)$$

is injective.

(d) Hence prove that S^1 is not a retract of $\mathbb{D}^2$. [You may quote any fundamental groups that you need, without proof.]

SECTION B

Answer **THREE** of the FOUR questions.

B5.

(a) Suppose that $q\colon X \to Y$ is a surjection from a topological space X to a set Y. Define the *quotient topology* on Y determined by q. State the *universal property* of the quotient topology.

(b) Suppose that $f\colon X \to Z$ is a continuous surjection from a compact topological space X to a Hausdorff topological space Z. Define an equivalence relation $\sim$ on X so that f induces a bijection $F\colon X/{\sim} \to Z$ from the identification space $Y = X/{\sim}$ of this equivalence relation to Z. Prove that F is a homeomorphism. [State clearly any general results which you use.]

(c) Prove that the quotient space $[0,1] \times [0,1]/\sim$ with $(0,s) \sim (1,s)$ is homeomorphic to the cylinder $[0,1] \times S^1 \subset \mathbb{R}^3$.

B6.

(a) Define what is meant by a *compact subset* of a topological space and by a *compact topological space.*

(b) Prove that, if $f\colon X \to Y$ is a continuous function of topological spaces and $K \subset X$ is a compact subset, then $f(K)$ is a compact subset of Y.

(c) Given a non-compact topological space (X, τ) consider the set $X^* = X \sqcup \{\infty\}$ and the topology

$$\tau^* = \tau \cup \{(X \setminus C) \cup \{\infty\} \mid C \subset X \text{ compact}\}.$$

Show that (X^*, τ^*) is a compact topological space.

B7.

(a) Prove that, if the product $\sigma_0 * \tau_0$ of two paths σ_0 and τ_0 in a topological space X is defined and the paths σ_1 and τ_1 are homotopic to σ_0 and τ_0 respectively, then the product $\sigma_1 * \tau_1$ is defined and is homotopic to $\sigma_0 * \tau_0$.

(b) Explain how a continuous function $f\colon X \to Y$ induces a homomorphism $f_*\colon \pi_1(X, x_0) \to \pi_1(X, f(x_0))$.You should indicate why f_* is well-defined and why it is a homomorphism.

(c) Prove that, for topological spaces X and Y with points $x_0 \in X$, $y_0 \in Y$, there is an isomorphism of groups

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

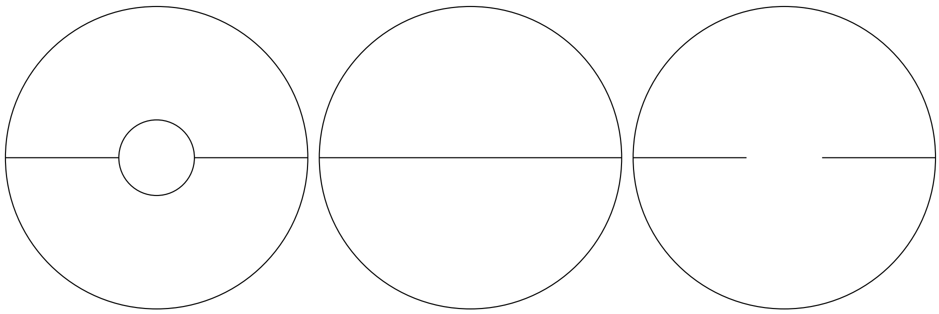
B8.

(a) Define what is meant by the path-components of a topological space. [You may assume the definition of a path and properties of paths.]

(b) Prove that a continuous map of topological spaces $f\colon X \to Y$ induces a map $f_*\colon \pi_0(X) \to \pi_0(Y)$ between the sets of path-components, taking care to prove that your function is well-defined. Prove that if f is a homeomorphism then f_* is a bijection.

(c) A pair of distinct points $\{p, q\}$ in a path-connected topological space X is called a cut-pair of type n when the subspace $X \setminus \{p, q\}$ has n path-components. Prove that a homeomorphism $f : X \to Y$ induces a bijection between the subsets of cut-pairs of type n.

(d) Hence show, using cut-pairs of order 3 or otherwise, that no two of the following subspaces of $\mathbb{R}^2$ with the usual topology are homeomorphic.



(i) (ii) (iii)

END OF EXAMINATION PAPER