SECTION A

A1.

(a) Define what is meant by a *topology* on a set X.

(b) Define what is meant by saying that a function $f\colon X \to Y$ between topological spaces is *continuous*. Define what is meant by saying that f is a *homeomorphism*.

(c) Prove that the closed disc $\mathbb{D}^2 = \{x \in \mathbb{R}^2 \mid |x| \leqslant 1\}$ with the usual topology is homeomorphic to the hemisphere $\{x = (x_1, x_2, x_3) \in S^2 \mid x_3 \geqslant 0\}$.

[Here S^2 denotes the unit sphere $\{x \in \mathbb{R}^3 \mid |x| = 1\}$ with the usual topology.]

[10 marks]

Solution

(a) Given a set X, a *topology* on X is a collection τ of subsets of X with the following properties:

(i) $\emptyset \in \tau$, $X \in \tau$;

(ii) the intersection of any two subsets in τ is in τ:

$$U_1, U_2 \in \tau \Rightarrow U_1 \cap U_2 \in \tau;$$

(iii) the union of any collection of subsets in τ is in τ:

$$U_\lambda \in \tau \text{ for all } \lambda \in \Lambda \Rightarrow \bigcup_{\lambda \in \Lambda} U_\lambda \in \tau.$$

[5 marks, bookwork]

(b) $f\colon X \to Y$ is *continuous* if

$$V \text{ is open in } Y \Rightarrow f^{-1}(V) \text{ is open in } X$$

[1 marks, bookwork]

A *homeomorphism* is a continuous bijection with continuous inverse.

[2 marks, bookwork]

(c) A homeomorphism $f\colon \{\, \mathbf{x} \in S^2 \mid x_3 \geqslant 0 \,\} \to D^2$ is given by $f(x_1, x_2, x_3) = (x_1, x_2)$ with inverse $f^{-1}(y_1, y_2) = (y_1, y_2, \sqrt{1 - y_1^2 - y_2^2})$.

[2 marks, question set]

[Total: 10 marks]

The question was generally well done. Some people didn't come up with the homeomorphisms in (c). In (b) sometimes only the (old) definition for the case of subspaces of $\mathbb{R}^n$ was given, but the question expilicitely asks for the case of (general) topological spaces (i.e. there is no notion of distance or ϵ-balls)

A2.

(a) Define what is meant by saying that a topological space X is *path-connected.*

(b) What is meant by saying the path-connectedness is a *topological property*?

(c) Prove that path-connectedness is a topological property.

(d) Prove that

$$\{(x_1, x_2) \in \mathbb{R}^2 \mid |(x_1, x_2 - 1)| \leqslant 1 \text{ or } |(x_1, x_2 + 1)| \leqslant 1\} \subset \mathbb{R}^2$$

(with the usual topology) is path-connected.

[10 marks]

Solution

(a) A *path* from x_0 to x_1 in X is a continuous function $\sigma\colon [0, 1] \to X$ with $\sigma(0) = x_0$ and $\sigma(1) = x_1$. X is said to be *path-connected* if, for each pair of points x_0, $x_1 \in X$, there is a path in X from x_0 to x_1.

[3 marks, bookwork]

(b) Saying that path-connectedness is a *topological property* means that, if $X \cong Y$ are homeomorphic topological spaces, then X is path connected if and only if Y is path-connected.

[1 marks, bookwork]

(c) To prove this, suppose that X is path-connected. Then, given two points y_0, $y_1 \in Y$ let x_0 , $x_1 \in X$ be points such that $f(x_i) = y_i$ (these points exist since f is a bijection). Since X is path-connected there is a path $\sigma\colon I \to X$ such that $\sigma(0) = x_0$ and $\sigma(1) = x_1$. Then $f \circ \sigma\colon I \to Y$ is a path in Y from y_0 to y_1 (since the composition of continuous maps is continuous). Hence, Y is path-connected. Conversely, if Y is path-connected then so is X by the same argument (interchanging the roles of X and Y).

[3 marks, bookwork]

(d) First observe that $0 \in X$. Now, there is a path from every point of X to 0, which implies path-connectedness by composition of paths. Indeed, for $x = (x_1, x_2) \in X$ consider the path $\sigma(t) = tx$. Assume first, that $|(x_1, x_2 - 1)| \leqslant 1$ then one has for $|\sigma(t) - (0, 1)|$

$$\begin{aligned} |tx - (0,1)| = |tx - t(0,1) - (1-t)(0,1)| &\leqslant |tx - t(0,1)| + |(1-t)(0,1)| \\ &= t \cdot |x - (0,1)| + (1-t) \\ &\leqslant 1. \end{aligned}$$

Hence $\sigma(t) \in X$ for $t \in [0, 1]$. Similarly for $|(x_1, x_2 + 1)| \leqslant 1$ one obtains $|\sigma(t) + (0, 1)| \leqslant 1$ and, hence, $\sigma(t) \in X$.

Hence, for to arbitrary points x, y a connecting path is given by

$$\tau(s) = \begin{cases} (1-2s)x & s \in [0, 1/2] \\ (2s-1)y & s \in [1/2, 1]. \end{cases}$$

[3 marks, new]

[Total: 10 marks]

Except from part (d) the question was generally done well. Sometimes in (c) people showed that for two points in Y of the form $f(x_0)$ and $f(x_1)$ there is a path. But you also need to refer to surjectivity of f to see that all elements in Y are of this form. For (d) some people constructed a path along the straight line between two points. This doesn't work here as the subset is not convex and, hence, doesn't contain the straight line between arbitrary points. A good point to start with was to sketch the subset in order to come up with a good guess for contructing a connecting path.

A3.

(a) Define what is meant by saying that a topological space is *Hausdorff*.

(b) Determine whether the set $S = \{a, b, c\}$ with topology $\tau = \{\emptyset, \{a, c\}, \{b\}, \{a, b, c\}\}$ is Hausdorff.

(c) Suppose that X and Y are topological spaces. Define the *product topology* on the Cartesian product $X \times Y$. [It is not necessary to prove that this is a topology.]

(d) Prove that if $\Delta \subset X \times X$ is closed in the product topology, then X is Hausdorff.

[10 marks]

Solution

(a) The topological space X is *Hausdorff* if, for each distinct pair of points x, $y \in X$, there exist open sets U and V in X such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

[2 marks, bookwork]

(b) This space is not Hausdorff because every open subset containing a also contains c and so open subsets as required cannot be found for $x = a$ and $y = c$.

[3 marks, bookwork]

(c) The product topology on $X \times Y$ has a basis

$$\{U \times V \mid U \text{ open in } X, V \text{ open in } Y\},$$

i.e. the open sets consist of all unions of such sets.

[3 marks, bookwork]

(d) Assume Δ is closed. Hence $X \times X \setminus \Delta$ is open. By definition of the product topology this means it is a union of open rectangles, i.e. sets of the form $U \times V \subset X \times X \setminus \Delta$ with U and V both open in X. Consider $x, y \in X$ with $x \neq y$ then (x, y) lies outside the diagonal. Hence, is has to be contained in such a set

$$U \times V \subset X \times X \setminus \Delta.$$

On the one hand this implies that $x \in U$ and $y \in V$. On the other hand $U \cap V = \emptyset$, since for $x \in U \cap V$ one would have $\Delta \ni (x, x) \in U \times V$. [2 marks, question set]

[Total: 10 marks]

Problems occurred in (c), where people forgot that open subsets not only arise as open rectangles, but also as unions of such rectangles. For (d) quite a few people didn't know that Δ was supposed to denote the diagonal, others did (indeed, the question was given in week 5 as an exercise with the solution being discussed in the tutorial). Nevertheless, in hindsight it would have been better to explicitly state the definition of Δ in the question. However, this issue was taken into account when marking the papers.

A4.

(a) Suppose that X_1 is a subspace of a topological space X. Define what is meant by saying that X_1 is a *retract* of X.

(b) Use the functorial properties of the fundamental group to prove that, if X_1 is a retract of X, then, for any $x_0 \in X_1$, the homomorphism induced by the inclusion map

$$i_*\colon \pi_1(X_1, x_0) \to \pi_1(X, x_0)$$

is injective.

(c) Hence prove that S^1 is not a retract of the closed disc $\mathbb{D}^2$.

[You may quote any fundamental groups that you need, without proof.]

[10 marks]

Solution

(a) $X_1 \subset X$ is a retract of X when there is a continuous map $r\colon X \to X_1$, such that $r(x) = x$ for $x \in X_1$. [3 marks, bookwork]

(b) By the functorial properties we have

$$r_* \circ i_* = (r \circ i)_* = (\mathrm{id}_{X_1})_* = \mathrm{id}_{\pi_1(X_1,x_0)}\colon \pi_1(X_1, x_0) \to \pi_1(X_1, x_0).$$

Since the composition of r_* and i_* is bijective r_* must be surjective and i_* must be injective.

[4 marks, bookwork]

(c) We have $\pi_1(S^1, x_0) = \mathbb{Z}$ and $\pi_1(D^2, x_0) = 1$, the trivial group. But there is not injective map $\mathbb{Z} \to \{1\}$. Hence, S^1 cannot be a retract of D^2. [3 marks, bookwork]

[Total: 10 marks]

In (a) sometimes it was stated that $i \circ r$ has to be the identity. Which clearly cannot be the case except if $X = X_1$.

SECTION B

B5.

(a) Suppose that $q\colon X \to Y$ is a surjection from a topological space X to a set Y . Define the *quotient topology* on Y determined by q. State the *universal property* of the quotient topology.

(b) Suppose that $f\colon X \to Z$ is a continuous surjection from a compact topological space X to a Hausdorff topological space Z . Define an equivalence relation $\sim$ on X so that f induces a bijection $F\colon X/\!\sim\; \to Z$ from the identification space $X/\!\sim$ of this equivalence relation to Z. Prove that F is a homeomorphism. [State clearly any general results which you use.]

(c) Prove that the quotient space $[0,1] \times [0,1]/\sim$ with $(0,s) \sim (1,s)$ is homeomorphic to the cylinder $[0,1] \times S^1 \subset \mathbb{R}^3$.

[15 marks]

Solution

(a) Given a topological space (X,τ) and a surjection $q\colon X \to Y$ the quotient topology on Y is given by

$$\{V \subset Y \mid q^{-1}(V) \in \tau\}.$$

The *universal property* of the quotient topology is: $f\colon Y \to Z$ to a topological space Z is continuous if and only if the composition $f \circ q\colon X \to Z$ is continuous.

[4 marks, bookwork]

(b) Given a continuous surjection $f\colon X \to Z$, define an equivalence relation on X by $x \sim x' \Leftrightarrow f(x) = f(x')$. Then we may define $F\colon X/\!\sim\to Z$ by $F([x]) = f(x)$. Since $[x] = [x'] \Leftrightarrow x \sim x' \Leftrightarrow f(x) = f(x')$ (by the definition of the equivalence relation), the function F is well-defined. Since $F([x]) = F([x']) \Leftrightarrow f(x) = f(x') \Leftrightarrow x \sim x'$ (by the definition of the equivalence relation) it follows that $[x] = [x']$ and F is injective. Since f is a surjection, $y = f(x)$ for some $x \in X$ and so $y = F([x])$. Hence F is a surjection. This shows that $F\colon X/\!\sim\; \to Z$ is a bijection. The map $F : X/\!\sim\; \to Z$ is continuous by the universal property since $F \circ q = f$ which is given as continuous, where $q\colon X \to X/\!\sim$ is the quotient map given by $q(x) = [x]$.

The space $X/\!\sim= q(X)$ is compact since it is the continuous image of a compact set. Hence F is a homeomorphism since it is a continuous bijection from a compact space to a Hausdorff space.

[7 marks, bookwork]

(c) To see this, define a surjection $f\colon I^2 \to I \times S^1$ by $f(x,y) = \big(y, \exp(2\pi i x)\big)$ where we think of S^1 as $\{\, z \in \mathbb{C} \mid |z| = 1 \,\}$ using the standard identification $\mathbb{C} \cong \mathbb{R}^2$. This function is continuous by the universal property of the product topology since the component functions are continuous. Now, $I \times S^1$ is Hausdorff (a subset of Euclidean space) and $I \times I$ is compact (a closed and bounded subset of Euclidean space). Now the result follows from (b).

[4 marks, bookwork]

[Total: 15 marks]

In (a) some people already referred to an equivalence relation $\sim$ which wasn't defined here and which doesn't play a role for defining the quotient topology for a surjection q. Sometimes open subsets where defined by $q^{-1}(V)$ is open $\Rightarrow V$ is open or V is open $\Rightarrow q^{-1}(V)$ is open, but here I insist on "if and only if". For (c) the most problems were caused by finding a suitable f. Also some people forgot to mention that $[0,1]\times[0,1]$ is compact and $[0,1]\times S^1$ is Haussdorff, which are necessary condition for being able to apply (b).

B6.

(a) Define what is meant by a *compact subset* of a topological space and by a *compact topological space.*

(b) Prove that, if $f\colon X \to Y$ is a continuous function of topological spaces and $K \subset X$ is a compact subset, then $f(K)$ is a compact subset of Y.

(c) Given a non-compact Hausdorff space (X,τ) consider the set $X^* = X \sqcup \{\infty\}$ and the topology

$$\tau^* = \tau \cup \{(X \setminus C) \cup \{\infty\} \mid C \subset X \text{ compact}\}.$$

Show that (X^*,τ^*) is compact.

[It is not necessary to prove that τ^* is a topology.]

[15 marks]

Solution

(a) $K \subset X$ is compact if each cover of K by open subsets of X has a finite subcover.

If X itself is a compact subset then X is a compact space.

[3 marks, bookwork]

(b) Suppose that $\mathcal{F}$ is an open cover for $f(K)$. Let $f^{-1}(\mathcal{F}) = \{f^{-1}(V) \mid V \in \mathcal{F}\}$. Then $f^{-1}(\mathcal{F})$ is an over cover for K since, given $a \in K$, $f(a) \in f(K)$ so that $f(a) \in V$ for some $V \in \mathcal{F}$. Hence $a \in f^{-1}(V)$ for some $V \in \mathcal{F}$.

Now, since K is compact, $f^{-1}(\mathcal{F})$ has a finite subcover for K, $\{\, f^{-1}(V_1), f^{-1}(V_2), \dots, f^{-1}(V_n)\,\}$. Thus, given $b \in f(K)$, $b = f(a)$ for some $a \in K$. Then $a \in f^{-1}(V_i)$ for some i, $1 \leqslant i \leqslant n$, so that $b = f(a) \in V_i$. Hence $\{\, V_1, V_2, \dots, V_n \,\}$ is a finite subcover of $\mathcal{F}$ for $f(K)$.

Hence $f(K)$ is compact.

[6 marks, bookwork]

(c) Consider an open cover $\mathcal{F}$ of X^*. In order to contain ∞ it has to include at least one open subset U_∞ of the form $X \setminus C \cup \{\infty\}$ where $C \subset X$ is compact. Now, $\mathcal{F}' = \{U \cap X \mid U \in \mathcal{F}\}$ is an open cover of X (since U and X are open in X^*) and hence of C.

By compactness of C a finite subcover $\{U_1 \cap X, \dots, U_m \cap X\} \subset \mathcal{F}$ suffices to cover C. But then one has the finite subcover $\{U_\infty, U_1, \dots, U_m\} \subset \mathcal{F}$.

[6 marks, exercise set]

[Total: 15 marks]

*A common mistake in (a) was the statement that a space is compact if all (open) subsets of it are compact. This cannot be true as you can see for example from $[0,1] \subset \mathbb{R}$. In (b) some people found some finite open cover (which is always possible – just take $\{X\}$) and argued that this implies compactness. Others argued that they found a finite subcover for a particular open cover, but one really needs that **every** open cover has a finite subcover. Only a few people successfully attempted (c). The main problem was to come up with an idea how to exploit the compactness of C in the definition of open subsets of the second kind.*

B7.

(a) Prove that, if the product $\sigma_0 * \tau_0$ of two paths σ_0 and τ_0 in a topological space X is defined and the paths σ_1 and τ_1 are homotopic to σ_0 and τ_0 respectively, then the product $\sigma_1 * \tau_1$ is defined and is homotopic to $\sigma_0 * \tau_0$.

(b) Explain how a continuous function $f\colon X \to Y$ induces a homomorphism $f_*\colon \pi_1(X, x_0) \to \pi_1(X, f(x_0))$. You should indicate why f_* is well-defined and why it is a homomorphism.

(c) Prove that, for topological spaces X and Y with points $x_0 \in X$, $y_0 \in Y$, there is an isomorphism of groups

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

[15 marks]

Solution

(a) Given homotopic paths $H\colon \sigma_0 \sim \sigma_1$ and $K\colon \tau_0 \sim \tau_1$ such that $\sigma_0 * \tau_0$ is defined. Then $_1(1) = \sigma_0(1) = \tau_0(0) = \tau_1(0)$ and so the product $\sigma_1 * \tau_1$ is defined.

Suppose that $H\colon \sigma_0 \sim \sigma_1$ and $K\colon \tau_0 \sim \tau_1$. Then we may define a homotopy $L\colon \sigma_0 * \tau_0 \sim \sigma_1 * \tau_1$ by

$$L(s,t) = \begin{cases} H(2s,t) & \text{for } 0 \leqslant s \leqslant 1/2 \text{ and } t \in I, \\ K(2s-1,t) & \text{for } 1/2 \leqslant s \leqslant 1 \text{ and } t \in I. \end{cases}$$

This is well defined since, for $s = 1/2$, $H(1,t) = x_1 = K(0,t)$. In addition, L is continuous by the Gluing Lemma since $[0,1/2] \times I$ and $[1/2,1] \times I$ are closed subsets of I^2

[5 marks, bookwork]

(b) The function f_* is defined by $f_*([\sigma]) = [f \circ \sigma]$. It is well-defined since, if $[\sigma_0] = [\sigma_1]$ then $\sigma_0 \sim \sigma_1$ and so there exists a homotopy $H\colon \sigma_0 \sim \sigma_1$. Then $f \circ H\colon I^2 \to Y$ gives a homotopy $f \circ \sigma_0 \sim f \circ \sigma_1$ and so $[f \circ \sigma_0] = [f \circ \sigma_1]$.

To see that f_* is a homomorphism suppose that $[\sigma]$, $[\tau] \in \pi_1(X, x_0)$. Then

$$f_*([\sigma][\tau]) = f_*([\sigma * \tau]) = [f \circ (\sigma * \tau)]$$

and

$$f_*([\sigma])f_*([\tau]) = [f \circ \sigma][f \circ \tau] = [(f \circ \sigma) * (f \circ \tau)]$$

and by writing out the formulae we see that $f \circ (\sigma * \tau) = (f \circ \sigma) * (f \circ \tau) \colon I \to Y$. Hence, $f_*([\sigma][\tau]) = f_*([\sigma])f_*([\tau])$.

[5 marks, bookwork]

(c) Let $p_1 \colon X \times Y \to X$ and $p_2 \colon X \times Y \to Y$ be the projection maps. The function

$$\pi_1\big(X \times Y, (x_0, y_0)\big) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

given by $\alpha \mapsto \big((p_1)_*(\alpha), (p_2)_*(\alpha)\big)$ is an isomorphism. To see this we write down the inverse. Given a loop σ_1 in X based at x_0 and a loop σ_2 in Y based at y_0 then we may define a loop σ in $X \times Y$ based at (x_0, y_0) by $\sigma(s) = \big(\sigma_1(s), \sigma_2(s)\big)$. Then $([\sigma_1], [\sigma_2]) \mapsto [\sigma]$ is well-defined and provides the necessary inverse. Hence the given function is an isomorphism of groups.

[5 marks, question set]

[Total: 15 marks]

In part (a) sometimes the reference to the continuity of L via Gluing Lemma was missing. There was a typo in part (b) it should read $f_* \colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$. This has been corrected during the exam. However, this shouldn't have prevented anybody from solving this part and most of you probably didn't even notice it. Only a few people succeeded with part (c).

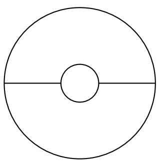
B8.

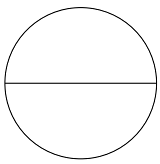
(a) Define what is meant by the *path-components* of a topological space. [You may assume the definition of a path and properties of paths.]

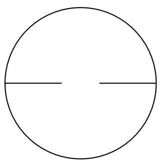
(b) Prove that a continuous map of topological spaces $f \colon X \to Y$ induces a map $f_* \colon \pi_0(X) \to \pi_0(Y)$ between the sets of path-components, taking care to prove that your function is well-defined. Prove that if f is a homeomorphism then f_* is a bijection.

(c) A pair of distinct points $\{p, q\}$ in a path-connected topological space X is called a cut-pair of type n when the subspace $X \setminus \{p, q\}$ has n path-components. Prove that a homeomorphism $f : X \to Y$ induces a bijection between the subsets of cut-pairs of type n for every $n \in \mathbb{N}$.

(d) Hence show, using cut-pairs of type 3 or otherwise, that no two of the following subspaces of $\mathbb{R}^2$ with the usual topology are homeomorphic.


(a)


(b)


(c)

[15 marks]

Solution

(a) Define an equivalence relation on X by $x \sim x'$ if and only if there is a path in X from x to x'. Then the path-components of X are the equivalence classes.

[2 marks, bookwork]

(b) Suppose that $f\colon X \to Y$ is a continuous map. Then this induces a function $f\colon \pi_0(X) \to \pi_0(Y)$ by $f([x]) = [f(x)]$. This is well-defined because $[x] = [x']$ implies that $x \sim x'$ so that there is a path $\sigma\colon [0,1] \to X$ in X from x to x'. Then $f \circ \sigma\colon [0,1] \to Y$ is a path in Y from $f(x)$ to $f(x')$ and so $[f(x)] = [f(x')]$.

[3 marks, bookwork]

If f is a homeomorphism then f_* is a bijection since the inverse $g = f^{-1}\colon Y \to X$ induces a function $g_* : \pi_0(Y) \to \pi_0(X)$ inverse to f_* since $g_*(f_*([x])) = [g(f(x))] = [x]$ and $f_*(g_*([y])) = [y]$.

[2 marks, bookwork]

(c) Suppose that $f\colon X \to Y$ is a homeomorphism and $\{p, q\}$ is a pair of distinct points in X. Then f induces a homeomorphism $X \setminus \{p, q\} \to Y \setminus \{f(p), f(q)\}$ and this induces a bijection $f_*\colon \pi_0(X \setminus \{p, q\}) \to \pi_0(Y \setminus \{f(p), f(q)\})$. Hence $\{p, q\}$ is a cut-pair of type n in X if and only if $\{f(p), f(q)\}$ is a cut-pair of type n in Y.

[3 marks, exercise set]

(d) In space (i) there are two cut-pairs of type 3 (the intersection points of the line segments and the inner or out circle respectively). In space (ii) there is a unique cut-pair of type 3 (the two points at the ends of the diameter). In space (iii) there are infinitely many cut-pairs of type 3 (picking two arbitrary points on the radial line segments). [5 marks, new]

[Total: 15 marks]

Part (a) and (b) were generally well done. Sometimes in the definition of f_* people mixed things up with the induced map on π_1 (instead of π_0) which is also denoted by f_*. In (c) people sometimes argued that there is a bijection between path-components of X and Y, but one needs this statement for $X \setminus \{p, q\}$ and $X \setminus \{f(p), f(q)\}$. For this it is not enough that the restriction of f is a (continuous) bijection but one really needs that it is a homeomorphism. For part (d) almost everyone used the correct approach but sometimes people didn't correctly identify cut pairs.

END OF EXAMINATION PAPER