

§0 Background material on sets and functions

The language of sets and functions is basic in topology and so it may be useful to provide a summary of results that students should already know. Most of this material will have been covered in the level 1 course units MATH10101 or MATH10111 *Sets, Numbers and Functions*. I shall not go through this material in the lectures but I shall refer to it from time to time.

Sets and subsets

0.1 Definition. A *set* is a well-defined collection of objects called the *elements*, *members* or *points* of the set. Usually a set is denoted by a upper case (capital) letter and the elements are denoted by lower case (small) letters. We write $x \in A$ to indicate that x is an element of the set X and to write $x \notin X$ to indicate that x is not an element of the set X .

We shall use the special symbols \mathbb{Z} to denote the set of integers, \mathbb{Q} to denote the set of rational numbers, \mathbb{R} to denote the set of real numbers, \mathbb{C} to denote the set of complex numbers, \mathbb{Z}^+ or \mathbb{N} to denote the set of positive integers (natural numbers) and \mathbb{Z}^{\geq} denotes the set of non-negative integers.

There are three basic ways of specifying a set:

- *listing the elements*: for example $A = \{1, 2, 3, 4\}$;
- *a conditional definition*: for example $A = \{n \in \mathbb{Z} \mid 1 \leq n \leq 4\}$ which gives the same set;
- *a constructive definition*: for example $B = \{n^2 \mid n \in \mathbb{Z} \text{ and } 1 \leq n \leq 4\} = \{1, 4, 9, 16\}$.

0.2 Definition. Two sets A and B are *equal*, written $A = B$, if they have precisely the same elements, i.e. $x \in A$ if and only if $x \in B$.

0.3 Definition. Given sets A and B we say that A is a *subset* of B , usually written $A \subset B$ (or sometimes $A \subseteq B$) if every element of A is also an element of B , i.e. $x \in A \Rightarrow x \in B$.

Thus $A = B$ if and only if $A \subset B$ and $B \subset A$.

The set of all subsets of a set X is called the *power set* of X and is denoted $\mathcal{P}(X)$.

It is particularly important to distinguish between an *element* of a set, $x \in X$, and a *subset* of a set, $A \subset X$; for example $1 \in \mathbb{Z}$ whereas $\{1\} \subset \mathbb{Z}$. A subset of the set X , $A \subset X$, is an element of its power set, $A \in \mathcal{P}(X)$: $A \subset X$ and $A \in \mathcal{P}(X)$ have precisely the same meaning.

A particularly useful set is the *empty set*, denoted \emptyset , which is the unique set with no elements at all, so $\emptyset = \{ \}$. Notice that the empty set is a subset of every set: for any set X , $\emptyset \subset X$ or $\emptyset \in \mathcal{P}(X)$.

There are standard notations for various subsets of the real line \mathbb{R} known as *intervals* with endpoints a and $b \in \mathbb{R}$ as follows:

- the *open interval* $(a, b) = \{ x \in \mathbb{R} \mid a < x < b \}$;
- the *closed interval* $[a, b] = \{ x \in \mathbb{R} \mid a \leq x \leq b \}$;
- the *right half-open interval* $[a, b) = \{ x \in \mathbb{R} \mid a \leq x < b \}$;
- the *left half-open interval* $(a, b] = \{ x \in \mathbb{R} \mid a < x \leq b \}$;
- the *open left-infinite interval* $(-\infty, a) = \{ x \in \mathbb{R} \mid x < a \}$;
- the *open right-infinite interval* $(a, \infty) = \{ x \in \mathbb{R} \mid x > a \}$;
- the *closed left-infinite interval* $(-\infty, a] = \{ x \in \mathbb{R} \mid x \leq a \}$;
- the *closed right-infinite interval* $[a, \infty) = \{ x \in \mathbb{R} \mid x \geq a \}$.

Notice that the use of ∞ and $-\infty$ in these definitions does not imply that these are elements in the set \mathbb{R} (they are not). Here these symbols only have meaning in the context of other notation (such as when you use these symbols when describing limits of infinite sequences or sums of infinite series in analysis or in the above infinite intervals).

In the names of these intervals ‘closed’ indicates that the end point is included in the set and ‘open’ indicates that it is not. These words are used in a much more general way in topology where ‘open subsets’ and ‘closed subsets’ will be defined (§2). Some care is required since the use of these words in these two contexts is not always consistent.

0.4 Definition. Given sets A and B we can form the following sets:

- *intersection:* $A \cap B = \{ x \mid x \in A \text{ and } x \in B \}$;
- *union:* $A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$;

- *difference*: $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$.

Two sets A and B are said to be *disjoint* if $A \cap B = \emptyset$ which means that they have no common elements.

We can extend the definitions of intersection and union to more than two sets. Given n sets A_i , $1 \leq i \leq n$, we can form their *intersection* $\bigcap_{i=1}^n A_i =$

$\{x \mid x \in A_i \text{ for all } i, 1 \leq i \leq n\}$ and their *union* $\bigcup_{i=1}^n A_i = \{x \mid x \in A_i \text{ for some } i, 1 \leq i \leq n\}$.

More generally, given a collection of sets A_λ where $\lambda \in \Lambda$ (we say that Λ is the *indexing set* for the collection of sets), we can form the following sets:

- *intersection*: $\bigcap_{\lambda \in \Lambda} A_\lambda = \{x \mid x \in A_\lambda \text{ for all } \lambda \in \Lambda\}$;
- *union*: $\bigcup_{\lambda \in \Lambda} A_\lambda = \{x \mid x \in A_\lambda \text{ for some } \lambda \in \Lambda\}$.

These operations on sets satisfy the following identities.

0.5 Proposition. Let A, B, C be any sets. Then we have the following identities.

- (i) *associativity*: $A \cap (B \cap C) = (A \cap B) \cap C = A \cap B \cap C$; $A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C$.
- (ii) *commutativity*: $A \cap B = B \cap A$; $A \cup B = B \cup A$.
- (iii) *distributivity*: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$; $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;
- (iv) *De Morgan laws*: $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$; $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

Proof. See Eccles¹, Theorem 6.3.4. □

We can extend these results to more general intersections and unions so that, for example, distributivity gives $A \cap \bigcup_{\lambda \in \Lambda} B_\lambda = \bigcup_{\lambda \in \Lambda} (A \cap B_\lambda)$ and the De

Morgan laws give $A \setminus \bigcap_{\lambda \in \Lambda} B_\lambda = \bigcup_{\lambda \in \Lambda} (A \setminus B_\lambda)$.

¹See the references at the end of these notes.

0.6 Definition. Given two sets A and B , the *Cartesian product* of A and B , denoted $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$: $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$.

Notice that two ordered pairs are equal, $(a_1, b_1) = (a_2, b_2)$, if and only if $a_1 = a_2$ and $b_1 = b_2$.

If $B = A$ then we write $A \times A$ as A^2 . When $A = \mathbb{R}$ this gives the Cartesian plane \mathbb{R}^2 . In this case it is necessary to be careful in distinguishing the ordered pair $(a, b) \in \mathbb{R}^2$ from the open interval $(a, b) \subset \mathbb{R}$. This is usually clear from the context. Unfortunately, in both cases the notation is a long established convention.

This can be extended to more than two sets: given n sets A_i , $1 \leq i \leq n$, their *Cartesian product*, denoted $\prod_{i=1}^n A_i$ or, more informally, $A_1 \times A_2 \times \cdots \times A_n$ is

the set of all ordered n -tuples (a_1, a_2, \dots, a_n) where $a_i \in A_i$. If all the sets are the same we write A^n (generalizing Cartesian n -space \mathbb{R}^n).

The product of an infinite collection of sets may also be defined. Given sets A_λ where $\lambda \in \Lambda$ then each element of the Cartesian product $\prod_{\lambda \in \Lambda} A_\lambda$ is

determined by a choice of element from each set, $a_\lambda \in A_\lambda$, so that $\prod_{\lambda \in \Lambda} A_\lambda = \{(a_\lambda \mid \lambda \in \Lambda) \mid a_\lambda \in A_\lambda\}$. The only significant use of infinite products will occur in the supplementary reading for MATH41051.

0.7 Proposition For sets A , B , C and D the following hold:

- (i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$;
- (ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$;
- (iii) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$;
- (iv) $(A \times B) \cup (C \times D) \subset (A \cup C) \times (B \cup D)$.

Proof. See Eccles, Proposition 7.7.4. □

Functions

0.8 Definition. A *function*, *map*, *map* or *mapping* from the set X to the set Y is an assignment of a unique element of Y to each element of X . If f is a function from X to Y then we write $f: X \rightarrow Y$ and denote the element of Y assigned to an element $x \in X$ by $f(x)$ writing $x \mapsto f(x)$. The element

$f(x) \in Y$ is called the *value* of f at $x \in X$. We say that x is a *pre-image* of y (under f). The set X is called the *domain* of f and the set Y is called the *codomain* of f .

A function is usually described by giving a formula for $f(x)$ or by giving a table of values.

Given functions $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ we define the *composition* $g \circ f: X \rightarrow Z$ to be the function given by $(g \circ f)(x) = g(f(x))$.

The *identity function*, $I_X: X \rightarrow X$ is given by $I_X(x) = x$ for all $x \in X$.

For $a \in Y$, the *constant function*, $c_a: X \rightarrow Y$ is given by $c_a(x) = a$ for all $x \in X$.

For a subset $A \subset X$ there is an inclusion function $i: A \rightarrow X$ given by $i(a) = a$ for all $a \in A$.

0.9 Definition. The *graph* of a function $f: X \rightarrow Y$ is defined to be the subset G_f of the Cartesian product $X \times Y$ given by

$$G_f = \{ (x, y) \in X \times Y \mid y = f(x) \} = \{ (x, f(x)) \mid x \in X \}.$$

Notice that the graph determines the function (this is familiar in the context of functions $\mathbb{R} \rightarrow \mathbb{R}$). In fact a function $X \rightarrow Y$ may be defined by its graph. Under this approach a function f is *defined* to be a subset of $G_f \subset X \times Y$ such that each element $x_0 \in X$ occurs as the first component of precisely one ordered pair in G_f . The *value* $f(x_0)$ of the function at x_0 is then *defined* to be the second component of the unique ordered pair $(x, y) \in G_f$ such that $x = x_0$. This avoids the rather vague term ‘assignment’ in Definition 0.8 and is the definition used in mathematical logic.

0.10 Definition. Suppose that $f: X \rightarrow Y$ is a function.

(a) The function f is an *injection* (or, equivalently, is *injective*) if the function takes a different value at each point of X , in symbols

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2) \text{ for all } x_1, x_2 \in X$$

or equivalently

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \text{ for all } x_1, x_2 \in X.$$

This means that each element of Y has at most one pre-image in X under f .

(b) The function f is a *surjection* (or is *surjective*) if each point of Y is a value of f at some point of X , in symbols

$$y \in Y \Rightarrow y = f(x) \text{ for some } x \in X.$$

This means that each element of Y has at least one pre-image in X under f .

(c) The function f is a *bijection* (or is *bijjective*) if it is both an injection and a surjection. This means that each element of Y has precisely one pre-image in X under f .

(d) If the function f is a bijection then we define the *inverse function* $f^{-1}: Y \rightarrow X$ by setting $f^{-1}(y) \in X$ to be the unique pre-image of y under f i.e.

$$f^{-1}(y) = x \Leftrightarrow y = f(x).$$

0.11 Definition. A set X is *finite* if there exists a bijection $\mathbb{N}_k \rightarrow X$ for some $k \in \mathbb{Z}^{\geq}$ where $\mathbb{N}_k = \{i \in \mathbb{N} \mid 1 \leq i \leq k\}$ (so $N_k = \emptyset$ for $k = 0$). A set X is *countable* if there exists a bijection $\mathbb{Z}^+ \rightarrow X$.

0.12 Proposition. A function $f: X \rightarrow Y$ is a bijection if and only if there is a function $g: Y \rightarrow X$ such that $g \circ f = 1_X: X \rightarrow X$, the identity function on X , and $f \circ g = 1_Y: Y \rightarrow Y$, the identity function on Y . In this case $g = f^{-1}$ and $f = g^{-1}$.

Proof. See Eccles, Theorem 9.2.3. □

0.13 Definition. A function $f: X \rightarrow Y$ induces functions between the power sets of X and Y as follows.

(a) The function $f: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ is defined by $f(A) = \{f(x) \mid x \in A\}$ for $A \in \mathcal{P}(X)$ (i.e. $A \subset X$).

(b) The function $f^{-1}: \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ is defined by $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$ for $B \in \mathcal{P}(Y)$ (i.e. $B \subset Y$).

It is unfortunate that the first of these has the same name as the original function and the second has the same name as the inverse function (which may or may not exist) although the meaning is usually clear from the context: the function and its inverse are evaluated on *elements* whereas the functions defined here are evaluated on *subsets*.

0.14 Proposition. For a function $f: X \rightarrow Y$, subsets $A_1, A_2 \subset X$, and subsets $B_1, B_2 \subset Y$,

(i) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$;

(ii) $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$ but not necessarily equal;

(iii) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$;

- (iv) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$;
- (v) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$;
- (vi) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.

Proof. See Eccles, Exercise 9.7 and Problem II.20. □

Equivalence relations

0.15 Definition. A *relation* on a set X is a property which may or may not be satisfied by each ordered pair of elements of the set. If an ordered pair of elements (a, b) satisfies the property then we say that they are related and write $a \sim b$ to indicate this. We write $a \not\sim b$ to indicate that the ordered pair does not satisfy the property.

As for functions, there is a formal definition of a relation as a subset R of the Cartesian product $R \subset X^2$. Then given an ordered pair of elements $a, b \in X$, $a \sim b \Leftrightarrow (a, b) \in R$ and $a \not\sim b \Leftrightarrow (a, b) \notin R$.

An *equivalence relation* is a relation satisfying the following properties:

- (i) *the reflexive property:* for all $a \in X$, $a \sim a$;
- (ii) *the symmetric property:* for all a and $b \in X$, if $a \sim b$ then $b \sim a$;
- (iii) *the transitive property:* for all a, b and $c \in X$, if $a \sim b$ and $b \sim c$ then $a \sim c$.

0.16 Definition. A *partition* Π of a non-empty subset X is a subset $\Pi \subset \mathcal{P}(X)$ of the power set of x (in other words a set of subsets of X) such that:

- (i) all the subsets in Π are non-empty: i.e. $A \in \Pi \Rightarrow A \neq \emptyset$;
- (ii) all the subsets in Π are disjoint, i.e. if A_1 and A_2 are two different subsets in Π (i.e. $A_1 \neq A_2$) then $A_1 \cap A_2 = \emptyset$;
- (iii) the subsets in Π *cover* X , i.e. $X = \bigcup_{A \in \Pi} A = X$.

0.17 Proposition. Given a partition Π of a non-empty set X the relation on X defined by $a \sim b$ if and only if a and b belong to the same subset of the partition is an equivalence relation.

Proof. See Eccles, Proposition 22.2.1. □

0.18 Definition. Suppose that \sim is an equivalence relation on a non-empty set X . For each $a \in X$ we define the *equivalence class* of a to be the subset $[a] \subset X$ of elements which are equivalent to a , i.e. $[a] = \{x \in X \mid x \sim a\}$.

We denote the set of all equivalence classes by X/\sim , i.e. $X/\sim = \{[a] \mid a \in X\} \subset \mathcal{P}(X)$.

0.19 Theorem. Suppose that \sim is an equivalence relation on a non-empty set X . Then for $a, b \in X$,

- (i) $a \sim b \Leftrightarrow [a] = [b]$,
- (ii) $a \not\sim b \Leftrightarrow [a] \cap [b] = \emptyset$.

Hence, the set of equivalence classes X/\sim is a partition of X .

Proof. See Eccles, Proposition 22.3.2 and Corollary 22.3.3. □

0.20 Theorem. Suppose that $f: X \rightarrow Y$ is a surjection between non-empty sets. Then we may define an equivalence relation on X by

$$x \sim x' \Leftrightarrow f(x) = f(x').$$

Then the function f induces a bijection $F: X/\sim \rightarrow Y$ by

$$F([x]) = f(x) \text{ for } x \in X.$$

Proof. The function F is well-defined since, if $[x_1] = [x_2]$, $x_1 \sim x_2$ and so $f(x_1) = f(x_2)$ by the definition of the equivalence relation.

The function F is a surjection because f is a surjection: given $y \in Y$, there exists $x \in X$ such that $f(x) = y$ and so $F([x]) = y$.

To see that F is an injection, suppose $[x_1], [x_2] \in X/\sim$. Then $F([x_1]) = F([x_2]) \Rightarrow f(x_1) = f(x_2)$ (by the definition of F) $\Rightarrow x_1 \sim x_2$ (by the definition of \sim) $\Rightarrow [x_1] = [x_2]$.

Hence F is a bijection. □

Continuous functions on subsets of \mathbb{R}^n

0.21 Definition. Suppose that X and Y are subsets of Euclidean spaces. A function $\mathbf{f}: X \rightarrow Y$ is *continuous at* $\mathbf{x}_0 \in X$ if, for each real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that, for all $\mathbf{x} \in X$,

$$|\mathbf{x} - \mathbf{x}_0| < \delta \Rightarrow |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)| < \varepsilon.$$

The function \mathbf{f} is *continuous* if it is continuous at each $\mathbf{x}_0 \in X$.

This generalizes to subsets of \mathbb{R}^n the definition given in MATH20101 or MATH20111 (see Haggerty, Definition 5.2.2). The generalization to metric spaces was given in MATH20122. However, this is *not* the definition of a continuous function used in topology. The topological definition will be introduced in this lecture course (§2) and will be shown to be equivalent to the above definition for subsets of Euclidean space.

0.22 Remarks. We take for granted for following.

(a) The familiar functions of calculus (such as polynomial functions, rational functions, the exponential function, the logarithm, the trigonometric functions and their inverses) are continuous on their domains of definition. [These are discussed in analysis courses such as MATH20101 or MATH20111.]

(b) A function $\mathbf{f}: X \rightarrow \mathbb{R}^n$ given by

$$\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_n(\mathbf{x}))$$

is continuous if and only if the coordinate functions $f_i: X \rightarrow \mathbb{R}$ are all continuous. [This is proved in analysis courses but will be reproved in a more general form in this lecture course (§3).]

(c) If $\mathbf{f}: X \rightarrow Y$ and $\mathbf{g}: Y \rightarrow Z$ are continuous, then so is the composition $\mathbf{g} \circ \mathbf{f}: X \rightarrow Z$. [This is proved in analysis courses but will be reproved in a more general form in this lecture course (Problems 2).]

0.23 The Interval Theorem. Suppose that $f: [a, b] \rightarrow \mathbb{R}$ is a continuous function.

(a) **The boundedness property.** The function f is bounded by values of the function (i.e. there are real numbers c and $d \in [a, b]$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a, b]$ so that $f(c)$ is the minimum value of the function and $f(d)$ is the maximum value of the function).

(b) **The intermediate value property.** Every number γ between $f(a)$ and $f(b)$ is a value of f , i.e. there exists $t \in [0, 1]$ such that $f(t) = \gamma$.

These two results can be summed up in the statement that $f([a, b]) = [\alpha, \beta]$ for some $\alpha, \beta \in \mathbb{R}$.

Proof. See Haggerty, Theorems 5.3.1, 5.3.2, 5.3.3. □

References

- P.J. Eccles, *An Introduction to Mathematical Reasoning: numbers, sets and functions*, Cambridge University Press, 1997, chapters 6, 8, 9 and 22.
- R. Haggerty, *Fundamentals of Mathematical Analysis*, Second Edition, Addison-Wesley, 1993, chapter 5.