3 Constructing New Spaces

Subspaces

3.1 Remark. Given a topological space $X$, is there a natural way of putting a topology on a subset $X_1 \subset X$? One desirable property is the following since we do not expect the codomain of a map to affect whether or not it is continuous.

For all topological spaces $Y$, $f: Y \to X_1$ is continuous $\iff i \circ f: Y \to X$ is continuous.

Here $i: X_1 \to X$ denotes the inclusion map $i(x) = x$ for all $x \in X_1$.

There is just one topology on $X_1$ which has this property and this is known as the subspace topology. This property is known as the universal property of the subspace topology.

3.2 Definition. Given a topological space $(X, \tau)$ and a subset $X_1 \subset X$, then the subspace topology on $X_1$ (induced by $\tau$) is given by $\tau_1 = \{ U \cap X_1 \mid U \in \tau \}$, i.e. $V \subset X_1$ is open in $X_1$ if and only if $V = U \cap X_1$ where $U$ is some open set in $X$.

With this topology we say that $X_1$ is a subspace of $X$.

3.3 Proposition. Given a topological space $X$ and a subset $X_1 \subset X$, Definition 3.2 defines a topology on $X_1$. With this topology,

(a) the inclusion map $i: X_1 \to X$ is continuous;

(b) given a continuous function $f: X \to Y$ (where $Y$ is an topological space), the restriction $f|X_1 = f \circ i: X_1 \to Y$ is continuous;

(c) (the universal property) a function $f: Y \to X_1$ (where $Y$ is any topological space) is continuous if and only if $i \circ f: Y \to X$ is continuous.

Proof. To see that Definition 3.2 defines a topology we check the properties in Definition 2.11.
(i) $\emptyset$ and $X$ are open in $X$ and so $\emptyset \cap X_1 = X$ and $X \cap X_1 = X_1$ are open in $X_1$.

(ii) Given $V_1$ and $V_2$ open in $X_1$ then $V_i = U_i \cap X$ for $U_i$ open in $X$ ($i = 1, 2$). Hence $V_1 \cap V_2 = (U_1 \cap X_1) \cap (U_2 \cap X_1) = (U_1 \cap U_2) \cap X_1$ is open in $X_1$ since $U_1 \cap U_2$ is open in $X$.

(iii) Given $V_\lambda$ open in $X_1$ for $\lambda \in \Lambda$. Then $V_\lambda = U_\lambda \cap X_1$ where $U_\lambda$ is open in $X$ ($\lambda \in \Lambda$). Hence $\bigcup_{\lambda \in \Lambda} V_\lambda = \bigcup_{\lambda \in \Lambda} (U_\lambda \cap X_1) = (\bigcup_{\lambda \in \Lambda} U_\lambda) \cap X$ is open in $X_1$ since $\bigcup_{\lambda \in \Lambda} U_\lambda$ is open in $X$.

(a) Given $U$ open in $X$ then $i^{-1}(U) = U \cap X_1$ is open in $X_1$ and so $i$ is continuous.

(b) This follows from the fact the composition of continuous functions is continuous (Proposition 2.13).

(c) ‘$\Rightarrow$’: This follows from the fact that the composition of continuous functions is continuous.

‘$\Leftarrow$’: Suppose that $f : Y \to X_1$ is a function from a topological space $Y$ such that $i \circ f : Y \to X$ is continuous. Then given $V$ open in $X_1$, $V = U \cap X_1 = i^{-1}(U)$ for $U$ open in $X$. Thus $f^{-1}(V) = f^{-1}i^{-1}(U) = (i \circ f)^{-1}(U)$ is open in $Y$ since $i \circ f$ is continuous. Hence $f$ is continuous.

\[\square\]

3.4 Remark. (a) The subspace topology on $X \subset \mathbb{R}^n$ induced by the usual topology on $\mathbb{R}^n$ is the usual topology on $X$. [Exercise. Note that $B^X_{\varepsilon}(x) = B_{\varepsilon}(x) \cap X$ for $x \in X$ and $\varepsilon > 0$.]

(b) Given a subspace $X_1$ of a topological space $X$ it is not in general true that an open [closed] subset of $X_1$ is open [closed] in $X$. For example, $(1/2, 1]$ is open in $[0, 1]$ with the usual topology (since $(1/2, 1] = (1/2, 3/2) \cap [0, 1]$) but is not open in $\mathbb{R}$.

3.5 Proposition. Given a subspace $X_1$ of a topological space $X$, a subset $B \subset X_1$ is closed in $X_1$ if and only if $B = A \cap X_1$ where $A$ is some closed set in $X$.

Proof. Exercise. \[\square\]

3.6 Proposition. Suppose that $X_1$ is a subspace of a topological space $X$. Then all closed subsets of the subspace $X_1 \subset X$ are closed in $X$ if and only if $X_1$ is a closed subset of $X$. 

Proof. Exercise. \[\square\]
Proof. ‘⇒’: If all closed subsets of $X_1$ are closed in $X$ then, since $X_1$ is closed in $X_1$, it is closed in $X$.

‘⇐’: Suppose that $X_1$ is a closed subset of $X$. Then, given $B$ closed in $X_1$, $B = A \cap X_1$ where $A$ is closed in $X$ (by Proposition 3.5) and so $B$ is closed in $X$ (the intersection of two closed subsets).

3.7 Theorem (Gluing Lemma). Suppose that $X_1$ and $X_2$ are closed subspaces of a topological space $X$ such that $X = X_1 \cup X_2$. Suppose that $f_1: X_1 \to Y$ and $f_2: X_2 \to Y$ are continuous functions to a topological space $Y$ such that, for all $x \in X_1 \cap X_2$, $f_1(x) = f_2(x)$. Then the function $f: X \to Y$ defined by $f(x) = f_1(x)$ if $x \in X_1$, $f(x) = f_2(x)$ if $x \in X_2$ is well-defined and continuous.

Proof. $f$ is well-defined by the condition on $f_1$ and $f_2$ in the theorem. To see that $f$ is continuous it is sufficient, by Problems 2, Question 8, to prove that the inverse image of a closed set in $Y$ is closed in $X$. Given $A$ closed in $Y$, $f_j^{-1}(A)$ is closed in $X_j$ ($j = 1,2$) since $f_j$ is continuous and so, using Proposition 3.6, $f_j^{-1}(A)$ is closed in $X$ since $X_j$ is closed in $X$. It follows that $f^{-1}(A) = f_1^{-1}(A) \cup f_2^{-1}(A)$ is closed in $X$ and so $f$ is continuous.

3.8 Example. This result gives a justification for the continuity of the product of two paths $\sigma_1 \ast \sigma_2$ in a topological space $X$ (generalizing Definition 1.13(c) to topological spaces). For suppose that $\sigma_1: [0,1] \to X$ and $\sigma_2: [0,1] \to X$ are two paths in $X$ so that $\sigma_1(1) = \sigma_2(0)$. Then the product path $\sigma_1 \ast \sigma_2: [0,1] \to X$ is given by

$$
\sigma_1 \ast \sigma_2(s) = \begin{cases} 
\sigma_1(2s) & \text{for } 0 \leq s \leq 1/2, \\
\sigma_2(2s-1) & \text{for } 1/2 \leq s \leq 1.
\end{cases}
$$

(generalizing Definition 1.13(c)). Define

$$
f_1: [0,1/2] \to X \text{ to be the composition } [0,1/2] \xrightarrow{s \to 2s} [0,1] \xrightarrow{\sigma_1} X, \text{ and}
$$

$$
f_2: [1/2,1] \to X \text{ to be the composition } [1/2,1] \xrightarrow{s \to 2s-1} [0,1] \xrightarrow{\sigma_2} X.
$$

Then $f_1$ and $f_2$ are compositions of continuous functions and so continuous. We can apply the Gluing Lemma to these two functions since $[0,1/2]$ and $[1/2,1]$ are closed in $[0,1]$, the intersection $[0/1/2] \cap [1/2,1] = \{1/2\}$ and $f_1(1/2) = \sigma_1(1) = \sigma_2(0) = f_2(1/2)$. The well-defined continuous function given by the Lemma is $\sigma_1 \ast \sigma_2$. 

3
Product spaces

3.9 Remark. Given topological spaces $X_1$ and $X_2$, is there a natural way of putting a topology on the cartesian product $X_1 \times X_2$? One desirable property is the following since it would generalize the familiar property that a function into $\mathbb{R}^n$ is continuous if and only if the coordinate functions are continuous (see Remarks 0.22(b)).

For all topological spaces $Y$,
\[ f: Y \to X_1 \times X_2 \text{ is continuous} \iff p_i \circ f: Y \to X_i \text{ is continuous for } i = 1, 2. \]

Here $p_i: X_1 \times X_2 \to X_i$ denotes the projection map $p_i(x_1, x_2) = x_i$ for all $(x_1, x_2) \in X_1 \times X_2$.

There is just one topology on $X_1 \times X_2$ which has this property and this is known as the product topology. This property is known as the universal property of the product topology.

3.10 Definition. Given topological spaces $X_1$ and $X_2$. The product topology on $X_1 \times X_2$ is the topology with a basis \{ $U_1 \times U_2$ | $U_i$ open in $X_i$ for $i = 1, 2$ \}. With this topology $X_1 \times X_2$ is called the product of the spaces $X_1$ and $X_2$.

3.11 Proposition. Given topological spaces $X_1$ and $X_2$, the set given above is the basis for a topology on $X_1 \times X_2$. With this topology,
(a) the projection functions $p_i: X_1 \times X_2 \to X_i$ are continuous;
(b) (the universal property) a function $f: Y \to X_1 \times X_2$ (for $Y$ any topological space) is continuous if and only if the coordinate functions $p_i \circ f: Y \to X_i$ are continuous for $i = 1, 2$.

Proof. To see that the collection of subsets in Definition 3.10 is a basis for a topology on $X_1 \times X_2$ we use the result of Problems 2, Question 11. Given two basic open sets $U_1 \times U_2$ and $U'_1 \times U'_2$ in $X_1 \times X_2$ (i.e. $U_i$ and $U'_i$ are open in $X_i$ for $i = 1, 2$), then
\[ (U_1 \times U_2) \cap (U'_1 \times U'_2) = (U_1 \cap U'_1) \times (U_2 \cap U'_2) \]
(by Proposition 0.7(iii)) which is also a basic open set since $U_i \cap U'_i$ is open in $X_i$ for $i = 1, 2$. 

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(a) For $U$ open in $X_1$, $p_1^{-1}(U) = U \times X_2$ which is open in $X_1 \times X_2$. Hence $p_1$ is continuous. Similarly, $p_2$ is continuous.

(b) $\Rightarrow$: This follows from the continuity of a composition of continuous functions.

$\Leftarrow$: To prove that $f: Y \to X_1 \times X_2$ is continuous it is sufficient to prove that $f^{-1}(U_1 \times U_2)$ is open in $Y$ for basic open sets $U_1 \times U_2$ by Problems 2, Question 9. Given such a basic open set and a function $f: Y \to X_1 \times X_2$ such that the coordinate functions $p_1 \circ f$ and $p_2 \circ f$ are continuous, $(p_1 \circ f)^{-1}(U_1) = f^{-1}p_1^{-1}(U_1) = f^{-1}(U_1 \times X_2)$ is open in $Y$ and, similarly, $(p_2 \circ f)^{-1}(U_2) = f^{-1}(X_1 \times U_2)$ is open in $Y$. Hence, by taking the intersection of these open sets, $f^{-1}(U_1 \times X_2) \cap f^{-1}(X_1 \times U_2) = f^{-1}((U_1 \times X_2) \cap X_1 \times U_2) = f^{-1}(U_1 \times U_2)$ is open in $Y$ and so $f$ is continuous.

3.12 Remark. (a) In the same way we can define the product topology on any finite product $X_1 \times X_2 \times \cdots \times X_n$ of topological spaces: a basis is given by subsets of the form $U_1 \times U_2 \times \cdots \times U_n$ where $U_i$ is an open subset of $X_i$.

(b) For each point $x_2 \in X_2$, the subspace $X_1 \times \{x_2\}$ of the product space $X_1 \times X_2$ is homeomorphic to $X_1$.

To see this we prove that the obvious bijection $f: X_1 \to X_1 \times \{x_2\}$ given by $f(x) = (x, x_2)$ for $x \in X_1$ is a homeomorphism by using using the universal properties of the product topology and the subspace topology.

First of all, $f$ is continuous if and only if $i_1 = i \circ f: X_1 \to X_1 \times x_2 \to X_1 \times X_2$ is continuous (by the universal property of the subspace topology) and only if $p_1 \circ i_1 = I_{X_1}: X_1 \to X_1$ is continuous and $p_2 \circ i_1 = c_{x_2}: X_1 \to X_2$ is continuous (by the universal property of the product topology) and these maps are continuous by Examples 2.17(e) and (f). Hence $f$ is continuous.

Secondly, the function $f^{-1}: X_1 \times \{x_2\} \to X_1$ is the restriction of the projection map $p_1: X_1 \times X_2 \to X_1$ which is continuous by Proposition 3.11(a) and so is continuous by Proposition 3.3(b).

(c) Given subsets $Y_1 \subset X_1$ and $Y_2 \subset X_2$ of topological spaces $X_1$ and $X_2$ then $Y_1 \times Y_2$ may be topologized as (i) a subspace of the product space $X_1 \times X_2$, and (ii) the product of the subspaces $Y_1$ and $Y_2$. These two topologies are the same. [Exercise. Use the universal properties to show that the identity map $I_{Y_1 \times Y_2}: (Y_1 \times Y_2, \tau_1) \to (Y_1 \times Y_2, \tau_2)$ is
a homeomorphism where $\tau_1$ is the topology (i) and $\tau_2$ is the topology $\tau_2$.]

3.13 Example.  (a) Euclidean $n$-space $\mathbb{R}^n$ with the usual topology is homeomorphic to the product space $\mathbb{R}^{n-1} \times \mathbb{R}$ (with the usual topologies on $\mathbb{R}$ and $\mathbb{R}^{n-1}$. [Exercise.]

(b) If $X$ and $Y$ have the discrete topology then the product topology on $X \times Y$ is the discrete topology.

(c) The product space $[0, 1] \times S^1$ is called the cylinder.

(d) The product space $S^1 \times S^1$ is called the torus.

(e) The product space $D^2 \times S^1$ is called the solid torus.