Problems 4: Quotient Spaces

1. Prove Proposition 3.15.

2. Let $X = \{ x \in \mathbb{R}^2 \mid 1 \leq |x| \leq 2 \}$ be an annulus in $\mathbb{R}^2$ with the usual topology. Define a continuous bijection from the quotient space $X/S^1$ to unit disc $D^2$ in $\mathbb{R}^2$ with the usual topology.

   [It follows from a general result in §5 that such a map is a homeomorphism.]

3. Suppose that $\sim$ is the equivalence relation on $I^2$ generated by $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, y)$ for all $x, y \in I = [0, 1]$. Prove that there is a continuous bijection $I^2/\sim \to S^1 \times S^1$. [Example 3.20(c). It will follow from a general result in §5 that such a map is a homeomorphism.]

4. Define an equivalence relation on the product space $S^1 \times [-1, 1]$ by $(x, t) \sim (x', t')$ if and only if $(x, t) = (x', t')$ or $t = t' = 1$ or $t = t' = -1$. Define a continuous bijection from the quotient space $(S^1 \times [-1, 1])/\sim$ to $S^2$ with the usual topology.

   [It follows from a general result in §5 that such a map is a homeomorphism.]

5. Define an equivalence relation on $\mathbb{R}^{n+1} \setminus \{0\}$ by $x \sim \lambda x$ for all non-zero real numbers $\lambda$. Prove that the quotient space $(\mathbb{R}^{n+1} \setminus \{0\})/\sim$ is homeomorphic to $P^n$, real projective $n$-space (as defined in 3.21 in the notes).

   [This is the classical definition of projective $n$-space: as the set of lines through the origin in $\mathbb{R}^{n+1}$.]  

6. Let $f: S^2 \to \mathbb{R}^4$ be defined by

   $$ f(x_1, x_2, x_3) = (x_1^2 - x_2^2, x_1x_2, x_1x_3, x_2x_3). $$

   Prove that $f$ induces a continuous bijection $F: P^2 \to F(P^2) \subset \mathbb{R}^4$ where $F(P^2)$ has the usual topology as a subset of $\mathbb{R}^4$.

   [It is a little tricky to show that $f(x) = f(x') \Rightarrow x = \pm x'$ so if this causes
problems I suggest that you don’t spend too much time on it.
It follows from a general result in §5 that this map is a homeomorphism.
Such a homeomorphism is called an embedding of the projective plane in \( \mathbb{R}^4 \).
It can be shown that there is no embedding of the projective plane in \( \mathbb{R}^3 \).]

7. Let \( D^n_r(a) = \{ x \in \mathbb{R}^n \mid |x - a| \leq r \} \). Define an equivalence relation on the subset \( X = D_1^2(2, 0) \cup D_1^2(-2, 0) \) of \( \mathbb{R}^2 \) with the usual topology by \( (2, 0) + x \sim (-2, 0) + x \) for \( |x| = 1 \). Prove that there is a homeomorphism from the quotient space \( X/\sim \) to \( S^2 \) with the usual topology.
[This shows that gluing two discs together by their boundary circles gives a sphere. You can get a continuous function \( X \to S^2 \) which induces the homeomorphism by mapping one disc to the upper hemisphere (using the map of \( D^2 \to \{ x \in S^2 \mid x_2 \geq 0 \} \) asked for in Problems 1, Question 4) and the other to the lower hemisphere. You can prove that the inverse map is continuous using the Gluing Lemma.]

8. Define an equivalence relation on \( D^2 \) by \( x \sim x' \iff x = x' \) or \( (x, x') \in S^1 \) and \( x' = -x \). Prove that there is a continuous bijection from \( D^2/\sim \) to \( P^2 \).
[It follows from a general result in §5 that such a map is a homeomorphism.
To construct the bijection you may find it useful to make use of the map \( D^2 \to \{ x \in S^2 \mid x_2 \geq 0 \} \) asked for in Problems 1, Question 4.]