Problems 7: The Fundamental Group

1. (a) Given a path \( \sigma : I \to X \) from \( x_0 \) to \( x_1 \) in a topological space \( X \), prove that
\[
\sigma \ast \varepsilon_{x_1} \sim \sigma.
\]
[Proposition 6.8, second part]
(b) Given two homotopic paths \( \sigma_0 \sim \sigma_1 \) from \( x_0 \) to \( x_1 \) in a topological space \( X \), prove that \( \overline{\sigma}_0 \sim \overline{\sigma}_1 \). [Proposition 6.10]

2. Suppose that \( X \) is a convex subset of \( \mathbb{R}^n \) with the usual topology [see Problems 1, Question 7.] Prove that, all paths from \( x_0 \) to \( x_1 \in X \) are homotopic. Deduce that \( \pi_1(X) \cong I \), the trivial group.

3. Suppose that \( X \) is a path-connected space and \( x_0, x_1 \in X \). Prove that all paths from \( x_0 \) to \( x_1 \) are homotopic if and only if \( X \) is simply-connected.

4. Suppose that \( X \) is a path-connected topological space and \( x_0, x_1 \in X \). Prove that all paths \( \rho \) from \( x_0 \) to \( x_1 \) induce the same isomorphism \( u_\rho : \pi_1(X,x_0) \to \pi_1(X,x_1) \) if and only if the fundamental group \( \pi_1(X) \) is abelian.

5. Recall from the proof of Proposition 1.17 that a continuous function \( f : X \to Y \) of topological spaces induces a function \( f_* : \pi_0(X) \to \pi_0(Y) \) by \( f_*([x]) = [f(x)] \). Which of the following assertions are true in general? Give a proof or counterexample for each.
   (a) If \( f \) is surjective then \( f_* \) is surjective.
   (b) If \( f \) is injective then \( f_* \) is injective.
   (c) If \( f \) is bijective then \( f_* \) is bijective.