Autumn Semester 2013–2014

MATH31051 *Introduction to Topology*

January Examination Solutions and Feedback

A1. (a) A *topology* τ on a set X is a collection of subsets of X (the *open subsets* of the topology) such that

(i) $\emptyset$, $X \in \tau$;

(ii) $U_1, U_2 \in \tau \Rightarrow U_1 \cap U_2 \in \tau$;

(iii) $U_\lambda \in \tau$ for $\lambda \in \Lambda \Rightarrow \bigcup_{\lambda \in \Lambda} U_\lambda \in \tau$.

[3 marks, bookwork]

(b) $f\colon X \to Y$ is *continuous* when

$$V \text{ open in } Y \Rightarrow f^{-1}(V) \text{ open in } X.$$

f is a *homeomorphism* when it is a continuous bijection with a continuous inverse.

[3 marks, bookwork]

(c) Define $f\colon X \to Y$ by $f(\boldsymbol{x}) = (\boldsymbol{x}/|\boldsymbol{x}|, 2|\boldsymbol{x}|)$. The inverse of this map is given by $g(\boldsymbol{y}, t) = (t/2)\boldsymbol{y}$ and so f is a bijection. Both f and g are continuous since the component functions are standard continuous functions. Hence f is a homeomorphism and so X is homeomorphic to Y. [4 marks, unseen but similar to problems set]

[**Total marks 10**]

[*This question was reasonably well done. Quite a lot of people couldn't do the final part and a few people gave the epsilon-delta definition of continuity which doesn't make sense for a function between topological spaces.*]

A2. (a) Y has the quotient topology determined by q when $V \subset Y$ is open in Y if and only if $q^{-1}(V)$ is open in X. [2 marks, bookwork]

(b) Define $f\colon I^2 \to S^1 \times S^1$ by $f(x, y) = \big(\exp(2\pi i x), \exp(2\pi i y)\big)$ writing $S^1 \subset \mathbb{C}$ the unit circle in $\mathbb{C}$. Then,

$$f(x, y) = f(x', y') \Leftrightarrow (x, y) \sim (x', y'). \tag{1}$$

using the equivalence relation defined in the question since, for $x, x' \in I$, $\exp(2\pi i x) = \exp(2\pi i x') \Leftrightarrow x = x'$ or $x, x' \in \{0, 1\}$. The function f is continuous by the universal property of the product topology since the component functions are given by the exponential function which is continuous. Then f induces a function $F\colon I^2/\!\sim\, \to S^1 \times S^1$ by $F([x, y]) = f(x, y)$. This is a well-defined injection by statement (1). The function f is a surjection since $x \mapsto \exp(2\pi i x)$ gives a surjection $[0, 1] \to S^1$ and so F is a surjection. The function F is continuous since $F \circ q = f\colon I^2 \to I^2/\!\sim\, \to S^1 \times S^1$ is continuous, by the universal property of the quotient topology. Hence F is a continuous bijection.

[5 marks, problem set]

I^2 is compact (closed and bounded in $\mathbb{R}^2$) and $S^1 \times S^1$ is Hausdorff (product of subspaces of Euclidean space) and so F is a homeomorphism (since it is a continuous bijection from a compact space to a Hausdorff space). [3 marks, problem set]

[**Total marks 10**]

[*This was also reasonably well done.*]

A3. (a) Two paths σ_0 and $\sigma_1\colon I \to X$ from x to x' are homotopic if there is a continuous map (a homotopy) $H\colon I^2 \to X$ such that

$$\begin{aligned} H(s,0) &= \sigma_0(s), \\ H(s,1) &= \sigma_1(s), \\ H(0,t) &= x, \\ H(1,t) &= x' \end{aligned}$$

for all $s,\, t \in I$. [3 marks, bookwork]

(b) Given two paths σ and τ in X, the product $\sigma * \tau$ is defined when $\sigma(1) = \tau(0)$ and is given by

$$\sigma * \tau(s) = \begin{cases} \sigma(2s) & \text{for } 0 \leqslant s \leqslant 1/2, \\ \tau(2s-1) & \text{for } 1/2 \leqslant s \leqslant 1 \end{cases}$$

for $s \in I$. This function is well-defined (since $\sigma(1) = \tau(0)$) and is continuous by the Gluing Lemma (since $[0,1/2]$ and $[1/2,1]$ are closed in $[0,1]$). [3 marks, bookwork]

(c) The *fundamental group* of X based at x_0, $\pi_1(X,x_0)$, is the set of homotopy classes of loops in X based at x_0 (i.e. paths in X from x_0 to x_0) with multiplication given by $[\sigma]\cdot[\tau] = [\sigma * \tau]$.

The product is well-defined since, if $\sigma_0 \sim \sigma_1$ and $\tau_0 \sim \tau_1$ are pairs of homotopic loops in X based at x_0 then the loops $\sigma_0 * \tau_0$ and $\sigma_1 * \tau_1$ are homotopic.

[4 marks, bookwork]

[**Total marks 10**]

[*Some people got the paths the wrong way round in the definition of the product of two paths. Some people didn't read the last part of the question carefully: you were asked to state what needed to be proved in order to show that the product was well-defined. You didn't need to go on and prove this statement.*]

A4. (a) The function $f_*\colon \pi_1(X,x_0) \to \pi_1\big(Y, f(x_0)\big)$ is defined by $f_*[\sigma] = [f \circ \sigma]$ where σ is a loop in X based at x_0.

The function f_* is well-defined since, if $H\colon I^2 \to X$ is a homotopy between two loops σ_0 and σ_1 in X based at x_0 then $f \circ H$ is a homotopy between $f \circ \sigma_0$ and $f \circ \sigma_1$.

The function f_* is a homomorphism since $f_*([\sigma]\cdot[\tau]) = [f \circ (\sigma * \tau)] = [(f\circ\sigma) * (f \circ \tau)] = f_*([\sigma]) \cdot f_*([\tau])$.

[4 marks, bookwork]

(b) The functorial properties of the fundamental group are that (i) the identity map $1\colon X \to X$ induces the identity map $1_* = 1\colon \pi_1(X,x_0)$ (immediate from the definition since $1 \circ \sigma = \sigma$), and (ii) if $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous maps then $(g \circ f)_* = g_* \circ f_*\colon \pi_1(X,x_0) \to \pi\big(Z, gf(x_0)\big)$ (which is also immediate since $(g\circ f)\circ\sigma = g \circ (f \circ \sigma)$).

Now if $f\colon X \to Y$ is a homeomorphism then $g = f^{-1}\colon Y \to X$ is a continuous map inducing a homomorphism $g_*\colon \pi_1\big(Y, f(x_0)\big) \to \pi_1\big(X, gf(x_0)\big) = \pi_1(X,x_0)$. Then $g_* \circ f_* = (g \circ f)_* = 1_* = 1$ (using the functorial properties) and similarly $f_* \circ g_* = 1$. Hence f_* is an isomorphism. [6 marks, bookwork]

[**Total marks 10**]

[*This was not so well done. Quite a lot of people didn't have the idea of the argument for part (b).*]

B5. (a) X is *path-connected* if, for each pair of points x, $x' \in X$ there is a path in X from x to x'. [2 marks, bookwork]

(b) Given y, $y' \in Y$, since $f\colon X \to Y$ is a surjection there are points x, $x' \in X$ such that $f(x) = y$, $f(x') = y'$. Since X is path-connected, there is a path $\gamma\colon [0,1] \to X$ from x to x'. Then $f \circ \gamma\colon [0,1] \to Y$ is continuous (composition of continuous functions is continuous) and gives a path in Y from y to y'. Hence Y is path-connected.

[5 marks, problem set]

(c) We may define an equivalence relation on a topological space X by $x \sim x'$ if and only if there is a path in X from x to x'. Then the equivalence classes are called the *path-components* of X. [2 marks, bookwork]

(d) For the given X, the path-components of X are $\{\boldsymbol{x} \in X \mid x_1 > 0\} = X^+$ and $\{\boldsymbol{x} \in X \mid x_1 < 0\} = X^-$.

To see that X^+ is path-connected observe that a continuous surjection $(-\pi, \pi) \to X^+$ is given by $t \mapsto (1 + \cos t, \sin t)$ and $(-\pi, \pi)$ is path-connected. Similarly, X^- is path-connected.

To see that there is no path γ in X from points of X^- to points of X^+ observe that if we had such a path then $p_1 \circ \gamma\colon [0,1] \to X \to \mathbb{R}$ would give a continuous function with $p_1 \circ \gamma(0) < 0$ and $p_1 \circ \gamma(1) > 0$ not taking the value 0 contradicting the Intermediate Value Theorem. [6 marks, unseen]

[**Total marks 15**]

[*The last bit of the question was not well done. In being asked to justify your answer you were being asked to provide a proper proof. So it is necessary to show that each of the path-components is path-connected and it is also necessary to show that there is no path between the path components. For this second result I think that you have to invoke the Intermediate Value Theorem.*]

B6. (a) X is Hausdorff when for each pair of distinct points x, $y \in X$, there are open subsets U and V in X such that $x \in U$, $y \in V$ and $U \cap V \neq \emptyset$. [2 marks, bookwork]

(b) This space is not Hausdorff because every open subset containing b also contains c and so open subsets as required cannot be found for $x = b$ and $y = c$.

[2 marks, similar to problems set]

(c) The subspace topology on $X_1 \subset X$ is given by $\{\, U \cap X_1 \mid U \text{ open in } X \,\}$.

[2 marks, bookwork]

(d) Suppose X is a Hausdorff space and x, $y \in X_1 \subset X$ such that $x \neq y$. Then, since X is Hausdorff there are disjoint open subsets U, V in X such that $x \in U$ and $y \in V$. Then $U_1 = U \cap X_1$ is an open subset in X_1 and $x \in U_1$, $V_1 = V \cap X_1$ is an open subset in X_2 and $y \in V_1$. Furthermore $U_1 \cap V_1 \subset U \cap V = \emptyset$ and so U_1 and V_1 are disjoint. Hence X_1 is Hausdorff. [4 marks, problem set]

(e) For each point $x \in X \setminus \{a\}$, since X is Hausdorff, there are disjoint open subsets U_x and V_x of X such that $a \in U_x$ and $x \in V_x$. Since the subsets are disjoint, $a \notin V_x$ and so $V_x \subset X \setminus \{a\}$. Hence $X \setminus \{a\} = \bigcup_{x \in X \setminus \{x\}} V_x$ and so, being a union of open subsets, is open in X. Hence $\{a\}$ is closed in X. [4 marks, problem set]

The set $\mathbb{R}$ with the cofinite topology as all finite sets closed and so, in particular, singleton sets are closed but it is not Hausdorff. [1 mark, standard example]

[**Total marks: 15**]

[*This was reasonably well done although some people get muddled about the definition of Hausdorff: you have to start with points and produce disjoint open sets.*]

B7. (a) Given homotopic paths $H\colon \sigma_0 \sim \sigma_1$ and $K\colon \tau_0 \sim \tau_1$ such that $\sigma_0 * \tau_0$ is defined. Then $\sigma_1(1) = \sigma_0(1) = \tau_0(0) = \tau_1(0)$ and so the product $\sigma_1 * \tau_1$ is defined. A homotopy $\sigma_0 * \tau_0 \sim \sigma_1 * \tau_1$ is given by $H * K\colon I^2 \to X$ where

$$H * K(s,t) = \begin{cases} H(2s,t) & \text{for } 0 \leqslant s \leqslant 1/2, \\ K(2s-1,t) & \text{for } 1/2 \leqslant s \leqslant 1 \end{cases}$$

for $s,\ t \in I$. This function is well-defined since $H(1,t) = \sigma_0(1) = \tau_0(0) = K(0,t)$ and is continuous by the Gluing Lemma. [5 marks, bookwork]

(b) From the data, the terminal point of σ is the same as the initial point of τ and the terminal point of τ is the same as the initial point of ρ and so the products in the question are defined. The required homotopy is given by

$$H(s,t) = \begin{cases} \sigma\big(4s/(1+t)\big) & \text{for } 0 \leqslant s \leqslant (1+t)/4, \\ \tau\big(4(s-(1+t)/4)\big) & \text{for } (1+t)/4 \leqslant s \leqslant (2+t)/4, \\ \rho\big((s-(2+t)/4)/(1-(2+t)/4\big) & \text{for } (2+t)/4 \leqslant s \leqslant 1. \end{cases}$$

This function is well-defined, and is continuous by the Gluing Lemma.

[5 marks, bookwork]

(c) Let $p_1\colon X \times Y \to X$ and $p_2\colon X \times Y \to Y$ be the projection maps. Then the function

$$\pi_1\big(X \times Y, (x_0, y_0)\big) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

given by $\alpha \mapsto \big((p_1)_*(\alpha), (p_2)_*(\alpha)\big)$ is a homomorphism by the result of Question A4(a). To see that it is an isomorphism we write down the inverse. Given a loop σ_1 in X based at x_0 and a loop σ_2 in Y based at y_0 then we may define a loop σ in $X \times Y$ based at (x_0, y_0) by $\sigma(s) = \big(\sigma_1(s), \sigma_2(s)\big)$. Then $([\sigma_1], [\sigma_2]) \mapsto [\sigma]$ is well-defined and provides the necessary inverse. [5 marks, problem set]

[**Total marks 15**]

[*The first two parts were reasonably well done. Most people were stumped by the third part which had been on a problem sheet.*]

B8. (a) Suppose that $\sigma\colon I \to S^1$ is a loop in S^1 based at 1. Then by the Path-Lifting Theorem, there is a unique path $\widetilde{\sigma}\colon I \to \mathbb{R}$ such that $p \circ \widetilde{\sigma} = \sigma$ and $\widetilde{\sigma}(0) = 0$. Since $p \circ \widetilde{\sigma}(1) = \sigma(1) = 1$, $\widetilde{\sigma}(1) \in \mathbb{Z}$. We define this integer to be the *degree* of the loop σ, written $\deg(\sigma)$. [3 marks, bookwork]

(b) By the Monodromy Theorem, if $\sigma_0 \sim \sigma_1$ are homotopic loops in S^1 based at 1 with lifts $\widetilde{\sigma}_0$ and $\widetilde{\sigma}_1$ as above, then $\widetilde{\sigma}_0(1) = \widetilde{\sigma}_1(1)$ and so the function $\phi\colon \pi_1(X, x_0) \to \mathbb{Z}$ given by $\phi([\sigma]) = \deg(\sigma)$ is well-defined. [2 marks, bookwork]

To see that ϕ is a homomorphism, let σ and τ be loops in S^1 based at 1 with lifts $\widetilde{\sigma}$ and $\widetilde{\tau}$ as above. Then we may define a lift for the loop $\sigma * \tau$ by

$$\rho(s) = \begin{cases} \widetilde{\sigma}(2s) & \text{for } 0 \leqslant s \leqslant 1/2, \\ \widetilde{\sigma}(1) + \widetilde{\tau}(2s-1) & \text{for } 1/2 \leqslant s \leqslant 1. \end{cases}$$

which is well-defined and continuous by the Gluing Lemma. Hence $\deg(\sigma * \tau) = \rho(1) = \widetilde{\sigma}(1) + \widetilde{\tau}(1)$ and so $\phi([\sigma] \cdot [\tau]) = \phi([\sigma]) + \phi([\tau])$. [4 marks, bookwork]

(c) The lift of σ is given by $\widetilde{\sigma}(s) = ns$ and so $\deg(\sigma) = \widetilde{\sigma}(1) = n$. Hence, since n is an arbitrary integer, ϕ is an epimorphism. [2 marks, bookwork]

(d) We need to show that $\ker(\phi) = \{1\}$. Suppose that $\phi([\sigma]) = 0$. Then $\deg(\sigma) = 0$ and so the unique lift of σ to $\mathbb{R}$ with $\widetilde{\sigma}(0) = 0$ has $\widetilde{\sigma}(1) = 0$ and so is a loop in $\mathbb{R}$ based at 0. Then $\widetilde{\sigma} \sim \varepsilon_0$, the constant loop at 0 by a homotopy $H\colon I^2 \to \mathbb{R}$ defined by $H(s,t) = (1-t)\widetilde{\sigma}(s)$. Then $p \circ H\colon p \circ \widetilde{\sigma} = \sigma \sim p \circ \varepsilon_0 = \varepsilon_1$ and so $[\sigma] = [\varepsilon_1] = 1 \in \pi_1(S^1, 1)$. Hence $\ker(\phi) = \{1\}$ and so ϕ is a monomorphism.

[4 marks, bookwork]
[**Total marks 15**]

[*This is a fairly solid piece of bookwork which you knew or you didn't!*]

C9. (a) $N \subset X$ is a *neighbourhood* of $x \in X$ when there is an open subset $U \subset X$ such that $x \in U \subset N$.

Given $x \in U$, an open subset, then $x \in U \subset U$ and so U is a neighbourhood of x.

Conversely, if U is a neighbourhood of each of its points then, for each $x \in U$, there is an open subset U_x such that $x \in U_x \subset U$. Then $U = \bigcup_{x \in U} U_x$ a union of open subsets and so U is open. [5 marks, problem set]

(b) The *interior*, A°, of a subset $A \subset X$ is defined by $x \in A^\circ$ when A is a neighbourhood of x.

Suppose that U is an open subset and $U \subset A$. Then, for each $x \in U$, $x \in U \subset A$ and so A is a neighbourhood of x. Hence, $x \in A^\circ$ and so $U \subset A^\circ$.

To see that A° is open suppose that $x \in A^\circ$. Then A is a neighbourhood of x and so $x \in U \subset A$ for some open subset U. But then, by the previous argument, $x \in U \subset A^\circ$ and so A° is a neighbourhood of x. Hence, A° is a neighbourhood of each of its points and so, by the result of (a), A° is open. [6 marks, problem set]

(c) (i) For each $x \in (0, 1)$, $x \in [x, 1) \subset (0, 1)$ and so $(0, 1)$ is a neighbourhood of x. Hence $(0, 1)^\circ = (0, 1)$.

(ii) $[0, 1)$ is open and so $[0, 1)^\circ = [0, 1)$.

(iii) By (i), $(0, 1)$ is open and $(0, 1) \subset (0, 1]$ and so, by the result of (b), $(0, 1) \subset (0, 1]^\circ$. However, $(0, 1]$ is not a neighbourhood of 1 since the basic open subsets containing 1 all have the form $[a, b)$ where $a \leqslant 1 < b$ which includes $1 + b/2 \notin (0, 1]$. So any open subset containing 1 also contains a point not in $(0, 1]$. Hence $(0, 1]^\circ = (0, 1)$.

(iv) $[0, 1)$ is open and $[0, 1) \subset [0, 1]$ so that $[0, 1) \subset [0, 1]^\circ$. By a similar argument to (iii), $1 \notin [0, 1]^\circ$. Hence, $[0, 1]^\circ = [0, 1)$. [6 marks, unseen]

[**Total marks 17**]

[*Not all students had got through all of the level 4/6 supplementary reading but most had covered the first section. The last bit of the question which asked you to find the interiors of four sets was not so well done. Anyone just wrote down for answers without any explanation got very little credit. It is not meant to be a guessing game! The only time it is acceptable to write down an answer without explanation is if a question asks you to 'state without proof' or similar words.*]

C10. (a) X is a G-space if there is a G-action on X, i.e. a map $G \times X \to X$ written $(g, x) = g \cdot x$ such that $1 \cdot x = x$ and $g \cdot (h \cdot x) = (gh) \cdot x$ for all g, $h \in G$ and $x \in X$, such that the map $\theta_g \colon X \to X$ given by $\theta_g(x) = g \cdot x$ is continuous.

To see that θ_g is a homeomorphism we note that $\theta_{g^{-1}}$ is a continuous inverse since $\theta_g \theta_{g^{-1}}(x) = g \cdot (g^{-1} \cdot x) = (gg^{-1}) \cdot x = 1 \cdot x = x$ and similarly $\theta_{g^{-1}} \theta_g = 1 \colon X \to X$.

[5 marks, problem set]

(b) Given a G-space X, we may define an equivalence relation on X by $x \sim x' \Leftrightarrow x' = g \cdot x$ for some $g \in G$. The conditions on the G-action guarantee that this is an equivalence relation. We define $X/G = X/\sim$ with the quotient topology induced from X by the quotient map $q \colon X \to X/\sim$ given by $q(x) = [x]$, the equivalence class of x.

For a subset $U \subset X$, $x \in q^{-1}q(U) \Leftrightarrow q(x) \in q(U) \Leftrightarrow q(x) = q(x')$ where $x' \in U \Leftrightarrow x = gx'$ where $x' \in U$ and $g \in G \Leftrightarrow x \in \theta_g(U)$ where $g \in G \Leftrightarrow x \in \bigcup_{g \in G} \theta_g(U)$. This proves the identity $q^{-1}q(U) = \bigcup_{g \in G} \theta_g(U)$.

Hence, given an open subset $U \subset X$, $q^{-1}q(U)$ is a union of open subsets $\theta_g(U)$ (since the functions θ_g are homeomorphisms) and so is open. Hence, by the definition of the quotient topology, $q(U)$ is open in X/G and so q is an open map.

[5 marks, problem set]

(c) Define $f \colon S^2 \to D^2$ by $f(x_1, x_2, x_3) = (x_1, x_2)$. Then $f(x_1, x_2, x_3) = f(x_1', x_2', x_3') \Leftrightarrow x_1' = x_1$ and $x_2' = x_2 \Leftrightarrow x_1' = x_1$ and $x_2' = x_2$ and $x_3' = \pm x_3$ since $x_3 = \pm\sqrt{1 - x_1^2 - x_3^2}$. Hence $f(\boldsymbol{x}) = f(\boldsymbol{x}') \Leftrightarrow \boldsymbol{x}' = g \cdot \boldsymbol{x}$ for $g \in \mathbb{Z}_2$. Hence f induces a continuous bijection $F \colon S^2/\mathbb{Z}_2 \to D^2$ by $F([x]) = f(x)$. It is well-defined and an injection by the above property of f. It is a surjection since f is a surjection. It is continuous by the universal property of the quotient topology since $F \circ q = f$ which is continuous.

The continuous bijection F is a homeomorphism by a general theorem about a continuous bijection from a compact space to a Hausdorff space since S^2 is compact (since it is a closed bounded subset of $\mathbb{R}^3$ with the usual topology) so that its continuous image $S^2/\mathbb{Z}_2$ is also compact, and D^2 is Hausdorff since it is a subspace of $\mathbb{R}^3$.

[5 marks, problem set]

[**Total marks 15**]

[*There weren't any real mistakes here. Some people were not on top of the material.*]

C11. (a) A continuous map $p\colon \widetilde{X} \to X$ is a *covering* if (i) p is a surjection; (ii) for each $x \in X$ there is an open neighbourhood V of x such that $p^{-1}(V) = \bigcup_{\lambda\in\Lambda} U_\lambda$, a disjoint union of open subsets of $\widetilde{X}$ such that the restriction map $p|U_\lambda \colon U_\lambda \to V$ is a homeomorphism for each $\lambda \in \Lambda$.[3 marks, bookwork]

(b) Suppose that X is a G-space for a group G. Then the action of G on X is said to be *properly discontinuous* if, for each $x \in X$, there is an open neighbourhood U of x such that $\theta_{g_1}(U) \cap \theta_{g_2}(U) = \emptyset$ for all $g_1 \neq g_2 \in G$. [2 marks, bookwork]

(c) The action of G on X is said to be *free* when $g \cdot x = g' \cdot x \Leftrightarrow g = g'$ for all $g, g' \in G$, $x \in X$.

Let $G = \{g_0 = 1,\, g_1,\, \ldots g_n\}$. Given $x \in X$, since X is Hausdorff, we can prove that there are open neighbourhoods U_i of $g_i \cdot x$ $(0 \leqslant i \leqslant n)$ such that $U_0 \cap U_i = \emptyset$ for $i = 1, 2, \ldots, n$. To do this notice that, since X is Hausdorff and for $1 \leqslant i \leqslant n$, $g_i \cdot x \neq x$, there are disjoint open subsets V_i and U_i such that $x \in V_i$ and $g_i \cdot x \in U_i$. Put $U_0 = \bigcap_{i=1}^n V_i$ then U_0, U_1, $\ldots U_n$ are open subsets as required.

Put $U = \bigcap_{i=0}^n \theta_{g_i}^{-1}(U_i)$ an open neighbourhood of x. To see that $\theta_{g_1}(U) \cap \theta_{g_2}(U) = \emptyset$ for all $g_1 \neq g_2 \in G$ suppose for contradiction that $x \in \theta_{g_1}(U) \cap \theta_{g_2}(U)$ where $g_1 \neq g_2$. Then $g_1^{-1} \cdot x \in U \cap \theta_{g_1^{-1}g_2}(U) \subset U_0 \cap U_i = \emptyset$ where $g_1^{-1}g_2 = g_i$ ($\neq g_0$ since $g_1 \neq g_2$), giving the necessary contradiction. Hence $\theta_{g_1}(U) \cap \theta_{g_2}(U) = \emptyset$ and so the action is properly discontinuous. [7 marks, part bookwork, part problem set]

(d) The quotient map $q\colon X \to X/G$ is continuous and a surjection by definition. Furthermore q is an open map by the result of Question C10(b). Given $x \in X$, let U be an open neighbourhood of x as given by proper discontinuity. Then $V = q(U)$ an open neighbourhood of $[x] \in X/G$ and $q^{-1}(V) = \bigcup_{g\in G} \theta_g(U)$, a disjoint union. Finally, the restriction of q gives a continuous open bijection $\theta_g(U) \to V$ and so a homeomorphism. Hence q is a covering. [6 marks, problem set]

[**Total marks 18**]

[*There weren't any real mistakes here. Some people were not on top of the material.*]